JEE MAIN ANSWER KEY & SOLUTIONS

SUBJECT :- MATHEMATICS
CLASS :- 12th **PAPER CODE :- CWT-9**
CHAPTER :- DEFINITE INTEGRATION

ANSWER KEY

1.	(D)	**2.**	(C)	**3.**	(D)	**4.**	(A)	**5.**	(A)	**6.**	(D)	**7.**	(A)
8.	(A)	**9.**	(D)	**10.**	(B)	**11.**	(B)	**12.**	(A)	**13.**	(A)	**14.**	(D)
15.	(D)	**16.**	(D)	**17.**	(B)	**18.**	(A)	**19.**	(D)	**20.**	(C)	**21.**	1
22.	2	**23.**	8	**24.**	3	**25.**	1	**26.**	5	**27.**	3	**28.**	1
29.	3	**30.**	4										

SOLUTIONS

1. (D)

Sol. $I = \int_{-\pi}^{\pi} (\cos px - \sin qx)^2 \, dx$

$I = \int_{-\pi}^{\pi} (\cos px + \sin qx)^2 \, dx$

(Using King)

$2I = 2\int_{-\pi}^{\pi} (\cos^2 px + \sin^2 qx) \, dx$

$I = \int_0^{\pi} (2\cos^2 px + 2\sin^2 qx) \, dx$

$= \int_0^{\pi} (1+\cos 2px) + (1-\cos 2qx) \, dx$

$= 2\pi$ (Both the integrals vanish) .]

2. (C)

Sol. Consider

$\lim_{n\to\infty} \left(1+\frac{t}{n+1}\right)^n \quad t \in (0,2) \quad (1^\infty)$

$= e^{\lim_{n\to\infty} n\left(\frac{t}{n+1}\right)} = e^t$

$\therefore \quad I = \int_0^2 e^t dt = e^2 - 1$ Ans.]

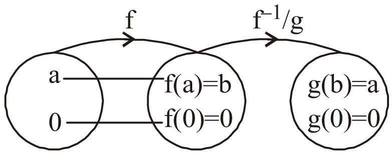
3. (D)

Sol. $y = f(x) \Rightarrow \quad x = g(y)$ and $dy = f'(x)\, dx$

$I = \int_0^a f(x)\,dx + \int_0^b g(y)\,dy$; $y = f(x) \Rightarrow x$

$= f^{-1}(y) = g(y)$

$= \int_0^a f(x)\,dx + \int_0^a x f'(x)\,dx$



$= \int_0^a (f(x) + x f'(x))dx = [x f(x)]_0^a$

$= a f(a) = ab$ Ans.]

4. (A)

Sol. $L = \lim_{n\to\infty} \frac{n}{(n!)^{1/n}} = \lim_{n\to\infty} \frac{n}{(1\cdot 2\cdot 3\cdot \ldots\ldots n)^{1/n}}$

$= \lim_{n\to\infty} \frac{1}{\left(\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\ldots\ldots\frac{n}{n}\right)^{1/n}}$

$\ln L = -\frac{1}{n}\left[\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) + \ln\left(\frac{3}{n}\right) + \ldots\ldots + \ln\left(\frac{n}{n}\right)\right]$

general term of $\ln L$ is

$T_r = -\frac{1}{n} \ln \frac{r}{n}$

$\therefore \quad S = \lim_{n\to\infty} -\frac{1}{n}\sum_{r=1}^{n} \ln\frac{r}{n} = -\int_0^1 \ln x \, dx$

$= -[x \ln x - x]_0^1 = -[(0-1)-(0)] = 1$

$\therefore$ Hence $\ln L = 1 \Rightarrow L = e$ **Ans.**]

5. (A)

Sol. Consider $\frac{d}{dx} \sin \cdot \frac{1}{\sin\left(\frac{1}{\sin x}\right)}$

$\cos\frac{1}{\sin\left(\frac{1}{\sin x}\right)}\cdot(-)\frac{1}{\sin^2\left(\frac{1}{\sin x}\right)}\cdot\cos\left(\frac{1}{\sin x}\right)\cdot(-1)\frac{1}{\sin^2 x}\cdot\cos x$

$= \cos\frac{1}{\sin\left(\frac{1}{\sin x}\right)}\cdot\cos\left(\frac{1}{\sin x}\right)\cdot\cos x\cdot\frac{1}{\sin^2 x\cdot\sin^2\left(\frac{1}{\sin x}\right)}$

hence the given integrand is

$$\frac{d}{dx}\sin\frac{1}{\sin\left(\frac{1}{\sin x}\right)}$$

$$\therefore \quad I = \int_{\pi/6}^{\pi/3}\left(\frac{d}{dx}\left(\sin\frac{1}{\sin\left(\frac{1}{\sin x}\right)}\right)\right)dx$$

$$= \left(\sin\frac{1}{\sin\left(\frac{1}{\sin x}\right)}\right)\Bigg|_{\pi/6}^{\pi/2} = \sin(\text{cosec } 1)$$

$-\sin(\text{cosec } 2)]$

6. (D)

Sol. Given $f(x) = \int_x^2 \frac{dy}{\sqrt{1+y^3}}$; $f(2) = 0$

also $f'(x) = -\frac{1}{\sqrt{1+x^3}}$

now let $I = \int_0^2 \underset{II}{x}\,\underset{I}{f(x)}\,dx$

$$= f(x)\cdot\frac{x^2}{2}\Bigg|_0^2 - \frac{1}{2}\int_0^2 f'(x)\,x^2\,dx$$

$$= 0 + \frac{1}{2}\int_0^2 \frac{x^2}{\sqrt{1+x^3}}dx$$

put $1 + x^3 = t^2 \Rightarrow 3x^2\,dx = 2t\,dt$

$$= \frac{1}{2}\cdot\frac{2}{3}\int_1^3\frac{t\,dt}{t} = \frac{1}{3}(3-1) = \frac{2}{3}$$ **Ans.**]

7. (A)

Sol. We have $f(x) = \alpha e^{2x} + \beta e^{x} - \gamma x$

Now $f(0) = -1 \Rightarrow \alpha + \beta = 1$ ....(1)

$f'(\ln 2) = 30 \Rightarrow 8\alpha + 2\beta - \gamma = 30$ ...(2)

and $\int_0^{\ln 4}(f(x)+\gamma x)\,dx$

$$= 24 \Rightarrow \int_0^{\ln 4}\left(\alpha e^{2x} + \beta e^{x}\right)dx = 24 \Rightarrow$$

$$\left(\frac{\alpha e^{2x}}{2} + \beta e^{x}\right)\Bigg|_0^{\ln 4} = 24 \Rightarrow 15\alpha + 6\beta = 48 \quad ....(3)$$

$\therefore$ On solving (1), (2), (3), we get $\alpha = 6$, $\beta = -7$, $\gamma = 4$

Hence $(\alpha + \beta + \gamma) = 6 - 7 + 4 = 3$ **Ans.**]

8. (A)

Sol. We have

$$f(x) = x^3 - 3x^2\int_{-1}^{1} y^2 f(y)dy + 3x\int_{-1}^{1} y f(y)dy$$

$\Rightarrow f(x) = x^3 + 3\alpha x^2 - 3\beta x,$

so $\alpha = \int_{-1}^{1} y^2(y^3 - 3\alpha y^2 + 3\beta y)\,dy$

$$= \left(\frac{y^6}{6} - 3\alpha\frac{y^5}{5} + 3\beta\frac{y^4}{4}\right)\Bigg|_{-1}^{1} = \frac{-6}{5}\alpha$$

$\Rightarrow \alpha = 0$

$\therefore f(x) = x^3 - 3\beta x.$

Now $\beta = \int_{-1}^{1} y(y^3 - 3\beta y)\,dy$

$$= 2\left(\frac{y^2}{5} - \beta y^3\right)\Bigg|_0^1 = \frac{2}{5} - 2\beta \Rightarrow \beta = \frac{2}{15}$$

$\therefore f(x) = x^3 - \frac{6}{15}x$

Hence $f(5) = 123$ **Ans.**]

9. (D)

Sol. $$\lim_{x\to 0}\frac{\int_0^x \sin t^2 dt}{x^3\frac{(1-\cos x)}{x^2}}$$ (using $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$)

$$= 2\lim_{x\to 0}\frac{\int_0^x \sin t^2 dt}{x^3}$$ (Using L'Hospital Rule)

$$= 2\lim_{x\to 0}\frac{\sin x^2}{3x^2} = \frac{2}{3}$$ Ans.]

10. (B)

Sol. I = $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} f(-x)\,dx$ (using K)

2I = $\int_{-1}^{1} \left(f(x)+f(-x)\right)dx = \int_{-1}^{1} (x^2)\,dx$

2I = $2\int_{0}^{1} (x^2)\,dx \Rightarrow$ I = $\int_{0}^{1} (x^2)\,dx$

= $\frac{1}{3}$ Ans.]

11. (B)

Sol. I = $\int_{0}^{\pi/2} \sqrt{\tan x}\,dx$ ....(1) ;

I = $\int_{0}^{\pi/2} \sqrt{\cot x}\,dx$ ....(2)

adding (1) and (2), we get

2I = $\int_{0}^{\pi/2} \left(\sqrt{\tan x} + \sqrt{\cot x}\right)dx$

= $\sqrt{2}\int_{0}^{\pi/2} \frac{\sin x + \cos x}{\sqrt{\sin 2x}}\,dx$

= $\sqrt{2}\int_{0}^{\pi/2} \frac{\sin x + \cos x}{\sqrt{1-(\sin x - \cos x)^2}}\,dx$

= $\sqrt{2}\int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} = 2\sqrt{2}\int_{0}^{1} \frac{dt}{\sqrt{1-t^2}} = \sqrt{2}\,\pi$

(where sin x – cos x = t)

$\therefore$ I = $\frac{\pi}{\sqrt{2}}$ Ans.]

12. (A)

Sol. I = $\int_{-1}^{1} \frac{x\tan^{-1}x\,dx}{(1+e^{\text{arc}\tan x})}$

Using King, I = $\int_{-1}^{1} \frac{x\tan^{-1}x}{(1+e^{-\tan^{-1}x})}\,dx$

= $\int_{-1}^{1} \frac{x\cdot\tan^{-1}x\cdot e^{\tan^{-1}x}}{(1+e^{\tan^{-1}x})}\,dx$

2I = $\int_{-1}^{1} \frac{x\tan^{-1}x}{(1+e^{\tan^{-1}x})}\left(1+e^{\tan^{-1}x}\right)dx$

= $\int_{-1}^{1} x\tan^{-1}x\,dx = 2\int_{0}^{1} x\tan^{-1}x\,dx$ (f is even)

I.B.P.

I = $\int_{0}^{1} x\tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}$ Ans.]

13. (A)

Sol. Let f (a) = $\int_{a}^{a^2} \frac{dx}{x+\sqrt{x}}$

f ' (a) = $\frac{2a}{a+a^2} - \frac{1}{a+\sqrt{a}} = 0 \Rightarrow 2a^2$

$+ 2a\sqrt{a} = a^2 + a \Rightarrow a^2 + 2a\sqrt{a} - a = 0$

$a + 2\sqrt{a} - 1 = 0 \Rightarrow (\sqrt{a}+1)^2 = 2$

$\Rightarrow \sqrt{a} = \sqrt{2} - 1 = \tan\frac{\pi}{8}$

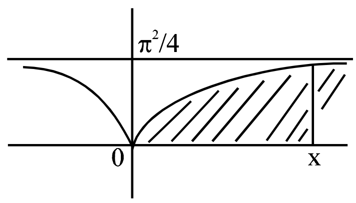
a = $(\sqrt{2}-1)^2 = \tan^2\left(\frac{\pi}{8}\right)$ Ans.]

14. (D)

Sol. graph of y = $(\tan^{-1}x)^2$.

Note that area enclosed by the curve and x-axis $\rightarrow\infty$.

Hence limit is of the form $\frac{\infty}{\infty}$ now using L Hospital rule,



hence $\lim_{x\to\infty} \frac{(\tan^{-1}x)^2}{x}\sqrt{1+x^2} = \frac{\pi^2}{4}$

Ans.]

15. (D)

Sol. Using King i.e. $x \to 2\alpha - x$

$$I = \int_0^{2\alpha} \frac{\sqrt[4]{\sin(\alpha+x)}}{\sqrt[4]{\sin(\alpha+x)} + \sqrt[4]{\sin(3\alpha-x)}} dx$$

$$\therefore \quad 2I = \int_0^{2\alpha} dx = 2\alpha \Rightarrow \quad I = \alpha \ \textbf{Ans.}]$$

16. (D)

Sol. $\int_a^0 3^{-x}(3^{-x}-2)\,dx \ge 0$

put $3^{-x} = t \Rightarrow 3^{-x}\ln 3\,dx = -dt$

$$\ln 3 \int_1^{3^{-a}} (t-2)\,dt \ge 0 \quad \Rightarrow$$

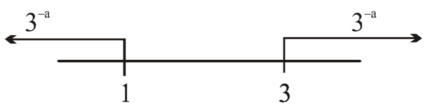
$$\left.\frac{t^2}{2} - 2t\right]_1^{3^{-a}} \ge 0$$

$$\left(\frac{3^{-2a}}{2} - 2.3^{-a}\right) - \left(\frac{1}{2} - 2\right) \ge 0$$

$3^{-2a} - 4\times3^{-a} + 3 > 0$

$(3^{-a}-3)(3^{-a}-1) > 0$

$3^{-a} > 3^1 \Rightarrow a < 1$



or $\quad 3^{-a} < 3^0 \Rightarrow a > 0$

Hence $\quad a \in (-\infty, -1] \cup [0, \infty)$]

17. (B)

Sol. $I = \int_{a-1}^{a} \frac{e^{-t}}{t-a-1} dt$ put $t = a-1+y$ (so that lower limit becomes zero)

$$\therefore I = \int_0^1 \frac{e^{1-a-y}}{y-2} dy \qquad \text{(now using king)}$$

$$I = \int_0^1 \frac{e^{1-a-1+y}}{1-y-2} dy = -e^{-a}\int_0^1 \frac{e^y}{1+y} dy = -e^{-a}$$

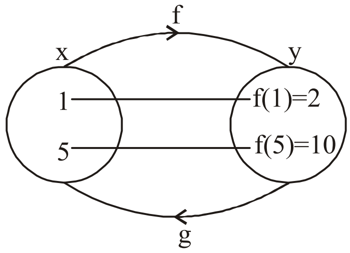
$A \Rightarrow$ (B)]

18. (A)

Sol. $y = f(x) \Rightarrow \quad x = f^{-1}(y) = g(y)$

$dy = f'(x)\,dx$

$$\therefore \quad I = \int_1^5 f(x)\,dx + \int_1^5 x f'(x)\,dx$$



where y is 2 then x = 1

y is 10 then x = 5

$$\therefore \quad I = \int_1^5 (f(x) + x f'(x))dx$$

$= x f(x)\big|_1^5 = 5f(5) - f(1) = 5 \cdot 10 - 2$

= 48 Ans.]

19. (D)

Sol. We have $S = \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\cos 4\left(\frac{k}{n}\right)$

$$= \int_0^1 \underbrace{(1-x)}_{I}\underbrace{\cos 4x}_{II}\,dx$$

$$= \left.(1-x)\frac{\sin 4x}{4}\right]_0^1 + \frac{1}{4}\int_0^1 \sin 4x\,dx$$

$$= 0 + \frac{1}{4}\int_0^1 \sin 4x\,dx = \left.\frac{-1}{16}\cos 4x\right]_0^1$$

$$= -\left(\frac{1}{16}\cos 4 - \frac{1}{16}\right) = \frac{1}{16}(1-\cos 4) \ \textbf{Ans.}]$$

20. (C)

Sol. $I = \int_0^{n\pi+V} |\cos x|\,dx$

$$= \underbrace{\int_0^{n\pi} |\cos x|\,dx}_{2n} + \underbrace{\int_{n\pi}^{n\pi+V} |\cos x|\,dx}_{I_1 \text{ (put } x=n\pi+t)}$$

So, $I_1 = \int_0^V |\cos t|\,dt$

$= \int_0^{\pi/2} \cos t\,dt - \int_{\pi/2}^{V} \cos x\,dx$

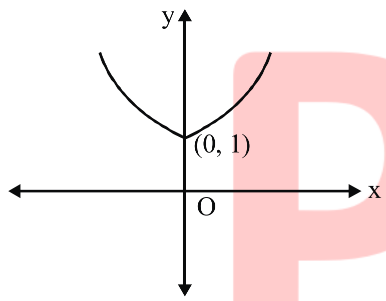
$= 1 - (\sin x)_{\pi/2}^{V} = 1 - \sin V + 1$

$\therefore \quad I = 2n + 2 - \sin V$]

21. 1

Sol. $\int_0^1 f(x)\,dx = \frac{11}{6} \Rightarrow 2\alpha + 3\beta + 6\gamma = 11$

Since, $\alpha, \beta, \gamma \in N \Rightarrow \alpha = \beta = \gamma = 1$ only possible



Hence, $f(x) = x^2 + x + 1$

$g(x) = |x|^2 + |x| + 1$

Clearly, g(x) is non-derivable at $x = 0$ only]

22. 2

Sol. $I_1 = \left.\frac{-x^{100}(1-x)^{51}}{51}\right|_0^1 + \int_0^1 \frac{100}{51} x^{99}(1-x)^{51}\,dx$

$= \frac{100}{51}\int_0^1 x^{99}(1-x)^{50}(1-x)\,dx$

$I_1 = \frac{100}{51}\left[\int_0^1 x^{99}(1-x)^{50}\,dx - \int_0^1 x^{100}(1-x)^{50}\,dx\right]$

$51 I_1 = 100 I_2 - 100 I_1$

$151 I_1 = 100 I_2$

$\frac{151 I_1}{50 I_2} = \frac{100}{50} = 2$ **Ans.**]

23. 8

Sol. Let $I = \int_0^1 {}^{207}C_7 \underbrace{x^{200}}_{II} \underbrace{(1-x)^7}_{I} dx$

$I = {}^{207}C_7 \left[\underbrace{\left.(1-x)^7 \frac{x^{201}}{201}\right|_0^1}_{\text{zero}} + \frac{7}{207}\int_0^1 (1-x)^6 \cdot x^{201} dx \right]$

$= {}^{207}C_7 \cdot \frac{7}{201}\int_0^1 (1-x)^6 x^{201} dx$

integrating by parts again 6 times more

$= {}^{207}C_7 \cdot \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{201\cdot202\cdot203\cdot204\cdot205\cdot206\cdot207}\int_0^1 x^{207} dx$

$= \frac{(207)!}{7!\,200!}\frac{7!}{201\cdot202\ldots\ldots207}\cdot\frac{1}{208} = \frac{1}{208}$

$= \frac{1}{k}$

$\Rightarrow k = 208$

Ans. $\frac{k}{26} = 8$]

24. 3

Sol. $f(x) = \left\{20 + \int_0^1 y f(y)\,dy\right\} x + \left\{49 + \int_0^1 y^2 f(y)\,dy\right\} x^2$

Let $f(x) = Ax + Bx^2$

then $A = 20 + \int_0^1 y f(y)\,dy$ and

$B = 49 + \int_0^1 y^2 f(y)\,dy$

$\Rightarrow \quad A = 20 + \int_0^1 (Ay + By^2) y\,dy$

$\Rightarrow \quad A = 20 + \frac{A}{3} + \frac{B}{4}$

$\Rightarrow \quad \frac{2A}{3} = 20 + \frac{B}{4}$

$B = \frac{8A}{3} - 80 \quad \ldots\ldots(1)$

Also, B = 49 + $\int_0^1 y^2(Ay + By^2)\,dy$

B = 49 + $\frac{A}{4} + \frac{B}{5}$

$\Rightarrow$ $\frac{4B}{5}$ = 49 + $\frac{A}{4}$

$\therefore$ B = $\frac{245}{4} + \frac{5A}{16}$ ......(2)

From (1) and (2)

$\frac{8A}{3}$ – 80 = $\frac{245}{4} + \frac{5A}{16}$

$\Rightarrow$ $\frac{8A}{3} - \frac{5A}{16} = \frac{565}{4}$ $\Rightarrow$ A = 60

$\therefore$ 20 + $\int_0^1 y\,f(y)$ = 60

$\therefore$ $\int_0^1 y\,f(y)$ = 40]

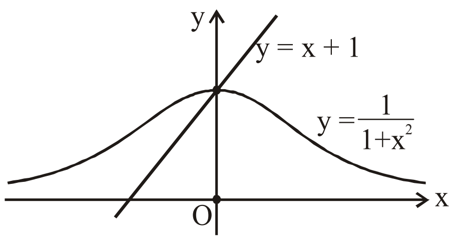
25. 1

Sol. $\int_{-1}^{0}\left(x - \frac{1}{1+x^2} - (x+1) + \frac{1}{1+x^2}\right)dx$

+ $\int_0^1\left(x - \frac{1}{1+x^2} + x + 1 - \frac{1}{1+x^2}\right)dx$

– $\int_{-1}^{0} 1\,dx$ + $\int_0^1\left(2x + 1 - \frac{2}{1+x^2}\right)dx$

$-(x)_{-1}^{0}$ + $\left(x^2 + x - 2\tan^{-1}x\right)_0^1$



– (0 – (–1)) + 2 – 2 tan^{-1} 1 = 1 – $\frac{\pi}{2}$

$\therefore$ 2(a + b) = 2$\left(1 - \frac{1}{2}\right)$ = 1]

26. 5

Sol. f (x) = $\int_0^1 \sin\left(\frac{\pi}{2}x\right) f(t)\,dt + \int_0^x x\,f(t)\,dt$

Let $\int_0^1 f(t)\,dt = \lambda$

f (x) = λ sin$\left(\frac{\pi}{2}x\right)$ + x$\int_0^x f(t)\,dt$

f (1) = λ + $\int_0^1 f(t)\,dt = 2\lambda$

Now, f '(x) = λ cos$\left(\frac{\pi}{2}x\right)\cdot\frac{\pi}{2}$ + x f (x)

+ $\int_0^x f(t)\,dt$

f '(1) = 0 + f (1) + λ = 3λ

$\therefore$ $\frac{f'(1)}{f(1)} = \frac{3\lambda}{2\lambda} = \frac{3}{2} = \frac{p}{q}$

$\Rightarrow$ p + q = 5 **Ans.**]

27. 3

Sol. g(x) = x^2 – $\left(f(2+t) - f(t)\right)$x – 4

Now, g(–x) = g(x)

$\Rightarrow$ f (2 + t) – f (t) = 0 $\Rightarrow$ f (t) = f(2 + t) $\forall$ t $\in$ R

$\int_0^{42} f(x)\,dx$ = 21$\int_0^2 f(x)\,dx$ = 21

$\left(\int_0^1 x\,e^{-x}\,dx + \int_1^2 \frac{-1}{e}(x-2)\,dx\right)$

= 21$\left(1 - \frac{2}{e} + \frac{1}{2e}\right)$ = 21$\left(1 - \frac{3}{2e}\right)$

= 21 – $\frac{63}{2e}$ $\equiv$ m – $\frac{n}{2e}$

$\therefore$ $\frac{n}{m} = \frac{63}{21}$ = 3 **Ans.**]

28. 1

Sol. $\int_0^1 \tan^{-1}\left(\dfrac{\dfrac{\tan x}{3} - \cot x}{1 + \dfrac{\tan x}{3}\cot x}\right) dx = \dfrac{3-\pi}{2} + \alpha$

$\Rightarrow \int_0^1 \tan^{-1}\left(\dfrac{\tan x}{3}\right) dx$

$- \int_0^1 \tan^{-1}(\cot x)\, dx = \dfrac{3-\pi}{2} + \alpha$

$\Rightarrow \int_0^1 \tan^{-1}\left(\dfrac{\tan x}{3}\right) dx - \int_0^1 \left(\dfrac{\pi}{2} - x\right) dx$

$= \dfrac{3-\pi}{2} + \alpha$

$\Rightarrow \int_0^1 \tan^{-1}\left(\dfrac{\tan x}{3}\right) dx - \left(\dfrac{\pi}{2} - \dfrac{1}{2}\right)$

$= \dfrac{3-\pi}{2} + \alpha \Rightarrow \int_0^1 \tan^{-1}\left(\dfrac{\tan x}{3}\right) dx - \alpha = 1.$

Ans.]

29. 3

Sol. $\because$ $x = f(x) + f(x)^7 \Rightarrow x = y + y^7$

$\Rightarrow f^{-1}(y) = y + y^7$

$\Rightarrow f^{-1}(x) = x^7 + x$

$\therefore$ Given integral = $\int_0^{\sqrt{2}} (x^7 + x)\, dx$

$= \left(\dfrac{x^8}{8} + \dfrac{x^2}{2}\right)_0^{\sqrt{2}}$ = (2 + 1) – 0 = 3 **Ans.**]

30. 4

Sol. $\lim_{x\to 0} \dfrac{\int_0^x f(t+2)\cdot(1-\cos(f(t)))dt}{\dfrac{x^3 \ln(1+x^2)}{x^2}} = \dfrac{1}{6}$

$\lim_{x\to 0} \dfrac{f(x+2)\cdot(1-\cos(f(x)))}{3x^2} = \dfrac{1}{6}$

$\lim_{x\to 0} \dfrac{f(x+2)\cdot(1-\cos(f(x)))}{3x^2(f(x))^2}\cdot(f(x))^2 = \dfrac{1}{6}$

$\dfrac{f(2)\cdot\dfrac{1}{2}\cdot\dfrac{1}{4}}{3} = \dfrac{1}{6} \Rightarrow f(2) = 4$ **Ans.**]