Topic :-INTEGRALS

1 **(d)**

Let $g_n(x) = 1 + x^2 + x^4 + \dots + x^{2n} = \dfrac{x^{2n+2} - 1}{x^2 - 1}$

$\therefore h_n(x) = g_n'(x) = \dfrac{2x(nx^{2n+2} - (n+1)x^{2n} + 1)}{(x^2 - 1)^2}$

Now, $f(x) = \lim\limits_{n \to \infty} h_n(x) = \dfrac{2x}{(x^2 - 1)^2}$

As $0 < x < 1$

Thus, $\int f(x)\,dx = \int \dfrac{2x}{(x^2 - 1)^2}\,dx$

$= -\dfrac{1}{x^2 - 1} = \dfrac{1}{1 - x^2}$

2 **(c)**

$\displaystyle\int \dfrac{dx}{\sin x - \cos x + \sqrt{2}}$

$= \displaystyle\int \dfrac{dx}{\sin x \dfrac{\sqrt{2}}{\sqrt{2}} - \cos x \dfrac{\sqrt{2}}{\sqrt{2}} + \sqrt{2}}$

$= \displaystyle\int \dfrac{dx}{\sqrt{2}\left(\sin x \sin\dfrac{\pi}{4} - \cos x \cos\dfrac{\pi}{4} + 1\right)}$

$= \dfrac{1}{\sqrt{2}} \displaystyle\int \dfrac{dx}{1 - \cos\left(x + \dfrac{\pi}{4}\right)}$

$= \dfrac{1}{\sqrt{2}} \displaystyle\int \dfrac{dx}{1 - \cos 2\left(\dfrac{x}{2} + \dfrac{\pi}{8}\right)}$

$= \dfrac{1}{\sqrt{2}} \displaystyle\int \dfrac{dx}{2\sin^2\left(\dfrac{x}{2} + \dfrac{\pi}{8}\right)}$

$= \dfrac{1}{2\sqrt{2}} \displaystyle\int \operatorname{cosec}^2\left(\dfrac{x}{2} + \dfrac{\pi}{8}\right)dx$

$$= -\frac{1}{2\sqrt{2}} \cdot \frac{-\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)}{\frac{1}{2}}+c=\frac{1}{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{\pi}{8}\right)+c$$

3   **(a)**

Let $I = \int_0^{\pi/2} \frac{\cos x}{1+\sin x} dx$

$= \left[\log(1+\sin x)\right]_0^{\pi/2}$

$= \log 2 - \log 1 = \log 2$

4   **(b)**

Let $I = \int_0^1 \frac{2\sin^{-1}\frac{x}{2}}{x} dx$

Put $\sin^{-1}\frac{x}{2} = t \Rightarrow \frac{x}{2} = \sin t$

$\Rightarrow x = 2\sin I$

$\Rightarrow dx = 2\cos t\, dt$

$\therefore I = \int_0^{\frac{\pi}{6}} \frac{2t}{2\sin t} \cdot 2\cos t\, dt$

$= \int_0^{\frac{\pi}{6}} \frac{2t}{\tan t} dt$

$= \int_0^{\frac{\pi}{6}} \frac{2x}{\tan x} dx$

5   **(a)**

Given ,$\int_2^e \left(\frac{1}{\log x} - \frac{1}{(\log x)^2}\right)dx = a + \frac{b}{\log 2}$

Put $\log x = z \Rightarrow x = e^z \Rightarrow dx = e^z dz$

$\therefore \int_2^e \left(\frac{1}{\log x} - \frac{1}{(\log x)^2}\right)dx = \int_{\log 2}^1 \left(\frac{1}{z} - \frac{1}{z^2}\right)e^z \ dz$

$= \int_{\log 2}^1 e^2\left(\frac{1}{z} + d\left(\frac{1}{z}\right)\right)dz$

$= \left[e^2 \cdot \frac{1}{z}\right]_{\log 2}^1 = e - \frac{2}{\log 2}$
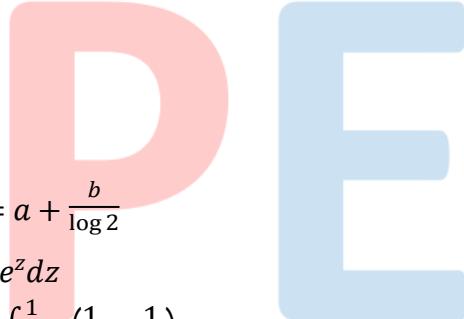
$\therefore a = e$ and $b = -2$

7   **(c)**

Let $I = \int_0^1 \frac{x\, dx}{[x+\sqrt{1-x^2}]\sqrt{1-x^2}}$

Put $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$\therefore I = \int_0^{\pi/2} \frac{\sin\theta\cos\theta d\theta}{(\sin\theta + \cos\theta)\cdot\cos\theta}$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta + \cos\theta} d\theta \ldots(i)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)} d\theta$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta + \cos\theta} d\theta \dots\text{(ii)}$$

On addingEqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \left(\frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta}\right) d\theta$$

$$= \int_0^{\pi/2} 1 d\theta = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

8   **(d)**

Let $I = \int_0^1 \cot^{-1}(1 - x + x^2) dx$

$$= \int_0^1 \tan^{-1}\left(\frac{1}{x^2 - x + 1}\right) dx$$

$$= \int_0^1 \tan^{-1}\left(\frac{x - (x - 1)}{1 + x(x - 1)}\right) dx$$

$$= \int_0^1 \tan^{-1} x\, dx - \int_0^1 \tan^{-1}(x - 1) dx$$

$$= \int_0^1 \tan^{-1} x\, dx - \int_0^1 \tan^{-1}(1 - x - 1) dx$$

$$= 2\int_0^1 \tan^{-1} x\, dx = 2\left[x \tan^{-1} x - \int \frac{x}{1 + x^2} dx\right]_0^1$$

$$= 2\left[x \tan^{-1} x - \frac{1}{2}\log(1 + x^2)\right]_0^1$$

$$= 2\left[\left(1 \tan^{-1} 1 - \frac{1}{2}\log 2\right) - \left(0 - \frac{1}{2}\log 1\right)\right]$$

$$= \frac{\pi}{2} - \log 2$$

9   **(a)**

Let $I = \int_8^{15} \frac{dx}{(x - 3)\sqrt{x + 1}}$

Put $x = \tan^2\theta \Rightarrow \theta = \tan^{-1}\sqrt{x}$

$dx = 2\tan\theta \sec^2\theta\, d\theta$

$$\therefore I = \int_{\tan^{-1}\sqrt{8}}^{\tan^{-1}\sqrt{15}} \frac{2\tan\theta \sec^2\theta}{(\tan^2\theta - 3)\sqrt{\tan^2\theta + 1}} d\theta$$

$$= \int_{\tan^{-1}\sqrt{8}}^{\tan^{-1}\sqrt{15}} \frac{2\tan\theta \sec^2\theta}{(\sec^2\theta - 1 - 3)\sec\theta} d\theta$$

$$= \int_{\tan^{-1}\sqrt{8}}^{\tan^{-1}\sqrt{15}} \frac{2\tan\theta\sec^2\theta}{(\sec^2\theta - 4)\sec\theta} d\theta$$

$$= \int_{\tan^{-1}\sqrt{8}}^{\tan^{-1}\sqrt{15}} \frac{2\tan\theta\sec\theta}{(\sec^2\theta - 4)} d\theta$$

$$= \left[\frac{1}{2}\log\frac{(\sec\theta - 2)}{(\sec\theta + 2)}\right]_{\tan^{-1}\sqrt{8}}^{\tan^{-1}\sqrt{15}} \left[\because \int \frac{1}{x^2 - a^2}dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{\sec\tan^{-1}\sqrt{15} - 2}{\sec\tan^{-1}\sqrt{15} + 2}\right) - \log\left(\frac{\sec\tan^{-1}\sqrt{8} - 2}{\sec\tan^{-1}\sqrt{8} + 2}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{\sec\sec^{-1}4 - 2}{\sec\sec^{-1}4 + 2}\right) - \log\left(\frac{\sec\sec^{-1}3 - 2}{\sec\sec^{-1}3 + 2}\right)\right]$$

$$= \frac{1}{2}\left[\log\frac{2}{6} - \log\frac{1}{5} = \frac{1}{2}\log\frac{5}{3}\right]$$

10   **(c)**

We have,

$$\Rightarrow I = \int \{1 + 2\tan x(\tan x + \sec x)\}^{1/2}dx$$

$$\Rightarrow I = \int \{1 + \tan^2 x + \tan^2 x + 2\tan x\sec x\}^{1/2}dx$$

$$\Rightarrow I = \int (\sec^2 x + \tan^2 x + 2\tan x\sec x)^{1/2}dx$$

$$\Rightarrow I = \int (\tan x + \sec x)dx$$

$$\Rightarrow I = \log\sec x + \log(\sec x + \tan x) + C$$

$$\Rightarrow I = \log\sec x(\sec x + \tan x) + C$$

11   **(a)**

We have,

$$I_1 = \int_0^\infty \frac{1}{1 + x^4}dx \text{ and } I_2 = \int_0^\infty \frac{x^2}{1 + x^4}dx$$

Putting $x = \frac{1}{t}$ in I_1, we get

$$I_1 = \int_\infty^0 \frac{t^4}{1 + t^4} \times -\frac{1}{t^2}dt = \int_0^\infty \frac{t^2}{t + t^4}dt = I_2 \Rightarrow \frac{I_1}{I_2} = 1$$

12   **(c)**

Let $I = \int \frac{\sin x}{3 + 4\cos^2 x}dx$

Put $\cos x = t \Rightarrow -\sin x\,dx = dt$

$$\therefore I = \int \frac{-dt}{3 + 4t^2} = \frac{-1}{4}\int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$\Rightarrow I = -\frac{1}{4 \cdot \frac{\sqrt{3}}{2}} \cdot \tan^{-1}\frac{t}{\left(\frac{\sqrt{3}}{2}\right)} + c$$

$$= -\frac{1}{2\sqrt{3}} \tan^{-1}\frac{2t}{\sqrt{3}} + c$$

$$\Rightarrow I = \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{2\cos x}{\sqrt{3}}\right) + c$$

13   **(c)**

Let $f(x) = e^{\cos^2 x} \cdot \cos^3(2n+1)x$

Then, $f(\pi - x) = e^{\cos^2(\pi - x)} \cdot \cos^3[(2n+1)\pi - (2n+1)x]$

$= -e^{\cos^2 x} \cdot \cos^3(2n+1)x$

$\Rightarrow f(\pi - x) = -f(x)$

$\therefore f(x)$ is an odd function.

Hence, $\int_0^{\pi} e^{\cos^2 x} \cdot \cos^3(2n+1)x \, dx = 0$

14   **(b)**

Given , $\frac{d}{dx}\{f(x)\} = \frac{1}{1+x^2}$

On integrating both sides, we get

$f(x) = \tan^{-1} x$

$\therefore \frac{d}{dx} f(x^3) = \frac{d}{dx}(\tan^{-1} x^3)$

$= \frac{1}{1+(x^3)^2} \cdot 3x^2$

$= \frac{3x^2}{1+x^6}$

15   **(d)**

Let $I = \int_0^{\pi}[\cot x]dx$ ...(i)

$$\Rightarrow I = \int_0^{\pi}[\cot(\pi - x)]dx = \int_0^{\pi}[-\cot x]dx \quad ...(ii)$$

On adding Eqs. (i) and (ii),

$$2I = \int_0^{\pi}[\cot x]dx + \int_0^{\pi}[-\cot x]dx$$

$$= \int_0^{\pi}(-1)dx$$

$[\because [x] + [-x] = -, \text{if} x \notin z \text{ and}[x] + [-x] = 0, \text{if} x \in z]$

$= [-x]_0^{\pi} = -\pi$

$\therefore I = -\frac{\pi}{2}$

16   **(c)**

Let $I = \int_{-3}^{2} (|x + 1| + |x + 2| + |x - 1|)\,dx$

Again, let $f(x) = |x + 1| + |x + 2| + |x - 1|$

$$= \begin{cases} -(x+1) - (x+2) - (x-1), & -3 < x \le -2 \\ -(x+1) + x + 2 - (x-1), & -2 < x \le -1 \\ 1 + x + x + 2 - (x-1) & -1 < x \le 0 \\ 1 + x + x + 2 - (x-1) & 0 \le x < 1 \\ 1 + x + x + 2 + x - 1 & 1 \le x < 2 \end{cases}$$
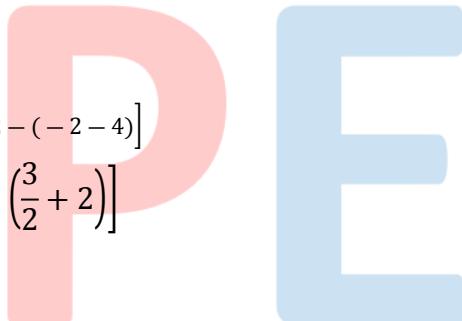
$$= \begin{cases} -3x - 2, & -3 < x \le -2 \\ -x + 2, & -2 < x \le -1 \\ x + 4, & -1 \le x < 1 \\ 3x + 2, & 1 \le x < 2 \end{cases}$$

$\therefore I = \displaystyle\int_{-3}^{-2} (-3 - 2)\,dx + \int_{-2}^{-1} (-x + 2)\,dx$

$+ \displaystyle\int_{-1}^{1} (x + 4)\,dx + \int_{1}^{2} (3x + 2)\,dx$

$= \left[-\dfrac{3x^2}{2} - 2x \right]_{-3}^{-2} + \left[-\dfrac{x^2}{2} + 2x \right]_{-2}^{-1}$

$+ \left[\dfrac{x^2}{2} + 4x \right]_{-1}^{1} + \left[\dfrac{3x^2}{2} + 2x \right]_{1}^{2}$

$= \left[-6 + 4 - \left(-\dfrac{27}{2} + 6 \right) \right] + \left[-\dfrac{1}{2} - 2 - (-2 - 4) \right]$

$+ \left[\dfrac{1}{2} + 4 - \left(\dfrac{1}{2} - 4 \right) \right]\left[6 + 4 - \left(\dfrac{3}{2} + 2 \right) \right]$

$= \dfrac{11}{2} + \dfrac{7}{2} + 8 + \dfrac{13}{2}$

$= \dfrac{31}{2} + 8 = \dfrac{47}{2}$

Alternate

Let $I = \int_{-3}^{2} \{|x + 1| + |x + 2| + |x - 1|\}\,dx$

$= \displaystyle\int_{-3}^{-1} |x + 1|\,dx + \int_{-1}^{2} |x + 1|\,dx + \int_{-3}^{-2} |x + 2|\,dx$

$+ \displaystyle\int_{-2}^{2} |x + 2|\,dx + \int_{-3}^{1} |x - 1|\,dx$

$+ \displaystyle\int_{1}^{2} |x - 1|\,dx$

$= -\displaystyle\int_{-3}^{-1} (x + 1)\,dx + \int_{-1}^{2} (x + 1)\,dx - \int_{-3}^{-2} (x + 2)\,dx$

$+ \displaystyle\int_{-2}^{2} (x + 2)\,dx - \int_{-3}^{1} (x - 1)\,dx + \int_{1}^{2} (x - 1)\,dx$

$$= -\left(\frac{x^2}{2} + x\right)_{-3}^{-1} + \left(\frac{x^2}{2} + x\right)_{-1}^{2} - \left(\frac{x^2}{2} + 2x\right)_{-3}^{-2}$$

$$+ \left(\frac{x^2}{2} + 2x\right)_{-2}^{2} - \left(\frac{x^2}{2} - x\right)_{-3}^{1} - \left(\frac{x^2}{2} - x\right)_{1}^{2}$$

$$= \frac{47}{2}$$

17 **(b)**

$$\int_0^3 |x^3 + x^2 + 3x| dx = \int_0^3 (x^3 + x^2 + 3x) dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} + \frac{3x^2}{2}\right]_0^3$$

$$= \frac{81}{4} + \frac{27}{3} + \frac{27}{2} = \frac{171}{4}$$

18 **(b)**

$$\int \frac{dx}{\sin x \cos x} = \int \frac{\cos x}{\sin x \cos^2 x} dx$$

$$= \int \frac{1}{\tan x} \sec^2 x \, dx = \log|\tan x| + c$$

19 **(b)**

Let $I = \int_0^{2n\pi} \left\{|\sin x| - \frac{1}{2}|\sin x|\right\} dx$

$$= \int_0^{2n\pi} \frac{1}{2} |\sin x| dx$$

$$= \frac{1}{2}\left[\int_0^{2\pi} |\sin x| dx + \int_{2\pi}^{4\pi} |\sin x| \, dx + \ldots\right.$$

$$+ \int_{2(n-1)\pi}^{2n\pi} |\sin x\} dx\Big]$$

Now, $I_1 = \int_0^{2\pi} |\sin x| dx$

$$= \int_0^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx$$

$$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi} = 4$$

$$\therefore I = \frac{1}{2}[4 + 4 + 4 + \ldots + n \text{ times}]$$

$$= \frac{1}{2}(4n) = 2n$$

$$I = \int_0^{2n\pi} \frac{1}{2} |\sin x| \, dx$$

Since $|\sin x|$ is periodic with period π.

$$\therefore I = 2n \int_0^\pi \frac{|\sin x| dx}{2}$$

$$I = n \int_0^\pi \sin x \, dx$$

$$= [-\cos x]_0^\pi$$

$$\Rightarrow I = 2n$$

Alternate

$$I = \int_0^{2n\pi} \frac{1}{2} |\sin x| dx$$

Since, $|\sin x|$ is periodic with period π.

$$\therefore I + 2n \int_0^\pi \frac{|\sin x|}{2} dx = n \int_0^\pi \sin x \, dx = n[-\cos x]_0^\pi$$

$$\Rightarrow I = 2n$$

20   **(d)**

Given, $\int_a^b x^3 \, dx = 0$ and $\int_a^b x^2 \, dx = \frac{2}{3}$

If we take $a = -1$ and $b = 1$, then it will satisfy the given integration.

ANSWER-KEY									
Q.	1	2	3	4	5	6	7	8	9	10
A.	D	C	A	B	A	A	C	D	A	C
Q.	11	12	13	14	15	16	17	18	19	20
A.	A	C	C	B	D	C	B	B	B	D