

Topic :-INTEGRALS

1 **(b)**

We have,

$$I = \int_0^{\pi} \log(1 + \cos x) dx$$

$$\Rightarrow I = \int_0^{\pi} \log\left(2 \cos^2 \frac{x}{2}\right) dx$$

$$\Rightarrow I = \int_0^{\pi} \left(\log 2 + 2 \log \cos \frac{x}{2}\right) dx$$

$$\Rightarrow I = \pi \log 2 + 2 \int_0^{\pi} \log \cos \frac{x}{2} dx$$

$$\Rightarrow I = \pi \log 2 + 2 \times 2 \int_0^{\pi/2} \log \cos t dt, \text{ where } t = \frac{x}{2} \text{ and } dx = 2dt$$

$$\Rightarrow I = \pi \log 2 + 4 \times -\frac{\pi}{2} \log 2 = -\pi \log 2$$

2 **(a)**

Putting $\tan^{-1} x = t$, we have

$$I = \int e^{\tan^{-1} x} \left(\frac{1+x+x^2}{1+x^2} \right) dx$$

$$\Rightarrow I = \int e^t (\tan t + \sec^2 t) dt = e^t \tan t + C = x e^{\tan^{-1} x} + C$$

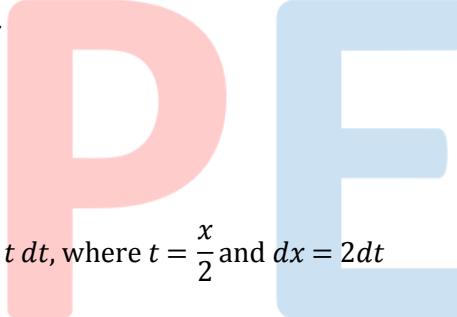
3 **(c)**

Given,

$$I_1 = \int_a^{\pi-a} x f(\sin x) dx$$

$$\text{and } I_2 = \int_a^{\pi-a} f(\sin x) dx$$

$$\begin{aligned} \text{Now, } I_1 &= \int_a^{\pi-a} x f(\sin x) dx \\ &= \int_a^{\pi-a} (\pi - x) f[\sin(\pi - x)] dx \end{aligned}$$



$$= \int_a^{\pi-a} (\pi - x) f(\sin x) dx$$

$$= \int_a^{\pi-a} \pi f(\sin x) dx - I_1$$

$$\Rightarrow 2I_1 = \pi I_2 \Rightarrow I_2 = \frac{2}{\pi} I_1$$

4 **(b)**

$$\int \cos^{-1} \left(\frac{1}{x} \right) dx = \int \sec^{-1} x \cdot 1 dx$$

$$= \sec^{-1} x \int dx - \int \left[\frac{d}{dx} \sec^{-1} x \int dx \right] dx$$

$$= x \sec^{-1} x - \int \frac{1}{x \sqrt{x^2 - 1}} x dx$$

$$= x \sec^{-1} x - \int \frac{1}{\sqrt{x^2 - 1}} dx$$

$$= x \sec^{-1} x - \cosh^{-1} x + c$$

5 **(d)**

$$\text{Let } I = \int_0^1 \frac{dx}{x + \sqrt{1-x^2}}$$

$$\text{Put } x = \sin \theta \Rightarrow dx = \cos \theta d\theta$$

$$I = \int_0^{\pi/2} \frac{\cos \theta d\theta}{\sin \theta + \cos \theta} \dots \text{(i)}$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos \left(\frac{\pi}{2} - \theta \right) d\theta}{\sin \left(\frac{\pi}{2} - \theta \right) + \cos \left(\frac{\pi}{2} - \theta \right)}$$

$$= \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta + \sin \theta} d\theta \dots \text{(ii)}$$

On adding Eqs.(i) and (ii), we get

$$2I = \int_0^{\pi/2} 1 d\theta \Rightarrow I = \frac{\pi}{4}$$

6 **(b)**

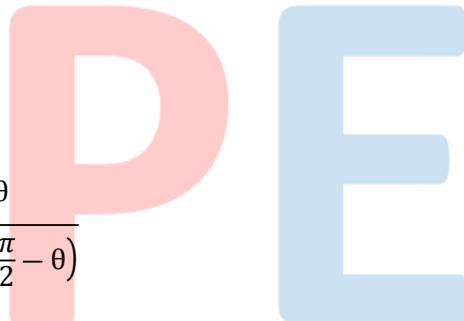
Let

$$I = \int_0^{\sqrt{n}} [x^2] dx$$

$$\Rightarrow I = \sum_{r=1}^n \int_{\sqrt{r-1}}^{\sqrt{r}} [x^2] dx$$

$$\Rightarrow I = \sum_{r=1}^n \int_{\sqrt{r-1}}^{\sqrt{r}} (r-1) dx$$

$$\Rightarrow I = \sum_{r=1}^n (r-1)(\sqrt{r} - \sqrt{r-1})$$



$$\Rightarrow I = (\sqrt{2} - \sqrt{1}) + 2(\sqrt{3} - \sqrt{2}) + 3(\sqrt{4} - \sqrt{3})$$

$$+ \dots + (n-1)(\sqrt{n} - \sqrt{n-1})$$

$$\Rightarrow I = n\sqrt{n} - (\sqrt{1} + \sqrt{2} + \dots + \sqrt{n}) = n\sqrt{n} - \sum_{r=1}^n \sqrt{r}$$

7 **(d)**

$$\text{Let } I = \int \frac{1}{x} (\log_e x) dx = \int \frac{1}{x(1 + \log_e x)} dx$$

$$\text{Put } \log_e x = t \Rightarrow \frac{1}{x} dx = dt$$

$$\therefore I = \int \frac{dt}{(1+t)} = \log_e(1+t) + c$$

$$= \log_e(1 + \log_e x) + c$$

8 **(a)**

$$I_1 - I_2 = \int_0^{\pi/2} (\cos \theta - \sin 2\theta) f(\sin \theta + \cos^2 \theta) d\theta$$

$$\text{Put } \sin \theta + \cos^2 \theta = t$$

$$\Rightarrow (\cos \theta - \sin 2\theta) d\theta = dt$$

$$\text{Then, } I_1 - I_2 = \int_1^1 f(t) dt = 0$$

$$\therefore I_1 = I_2$$

9 **(b)**

$$(1) I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x} dx$$

$$= \int_{-\pi/2}^{\pi/2} \sqrt{\cos x} \sin x dx$$

$$= 2 \int_0^{\pi/2} \sqrt{\cos x} \sin x dx$$

$$= -\frac{4}{3} [\cos^{3/2} x]_0^{\pi/2} = \frac{4}{3}$$

$$(2) I = \int_0^1 |x-1| dx + \int_1^4 |x-1| dx + \int_0^3 |x-3| dx + \int_3^4 |x-3| dx$$

$$= \int_0^1 -(x-1) dx + \int_1^4 (x-1) dx + \int_0^3 -(x-3) dx + \int_3^4 (x-3) dx$$

$$= 10$$

10 **(a)**

$$\text{Let } I = \int_{-\pi/4}^{\pi/4} \sin^{-4} x dx = \int_{-\pi/4}^{\pi/4} \cosec^{-4} x dx$$

$$= \int_{-\pi/4}^{\pi/4} (1 + \cot^2 x) \cosec^2 x dx$$

$$\text{Put } \cot x = t$$

$$\Rightarrow -\cosec^2 x dx = dt$$

$$\therefore I = - \int_{-1}^1 (1+t^2) dt = -2 \int_0^1 (1+t^2) dt$$



$$= -2 \left[t + \frac{t^3}{3} \right]_0^1 = -2 \left[1 + \frac{1}{3} \right] = -\frac{8}{3}$$

11 (d)

We have,

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\left(\int_0^x e^x dx \right)^2}{\int_0^x e^{2x^2} dx} &= \lim_{x \rightarrow \infty} \frac{(e^x - 1)^2}{\int_0^x e^{2x^2} dx} \\ &= \lim_{x \rightarrow \infty} \frac{2(e^x - 1)e^x}{\int_0^x e^{2x^2} dx} \quad [\text{Using L'Hospital's rule}] \\ &= 2 \lim_{x \rightarrow \infty} \frac{e^x - 1}{e^{2x^2-x}(4x-1)} \\ &= 2 \lim_{x \rightarrow \infty} \frac{1}{e^{2x^2-2x}(4x-1)} = 0 \end{aligned}$$

12 (a)

$$\text{Given, } \int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$$

$$\begin{aligned} \text{Now, } \frac{d}{dx} \int_{\sin x}^1 t^2 f(t) dt &= \frac{d}{dx} (1 - \sin x) \\ &\Rightarrow [1^2 f(1)] \cdot (0) - (\sin^2 x) \cdot f(\sin x) \cdot \cos x = -\cos x \end{aligned}$$

[by Leibnitz formula]

$$\Rightarrow \text{Put } \sin x = 1/\sqrt{3}$$

$$\therefore f\left(\frac{1}{\sqrt{3}}\right) = (\sqrt{3})^2 = 3$$

13 (b)

We have,

$$f(a-x) + f(a+x) = 0$$

$$\Rightarrow f(2a-x) + f(x) = 0 \quad [\text{On replacing } x \text{ by } x-a]$$

$$\Rightarrow f(2a-x) = -f(x)$$

$$\therefore \int_0^{2a} f(x) dx = \int_0^a \{f(x) + f(2a-x)\} dx$$

$$\therefore \int_0^{2a} f(x) dx = \int_0^a \{f(x) - f(x)\} dx = 0$$

14 (a)

$$\begin{aligned} \int \frac{dx}{\sin(x-a) \sin(x-b)} &= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a) \sin(x-b)} dx \\ &= \frac{1}{\sin(a-b)} \end{aligned}$$

$$\begin{aligned}
& \int \frac{\sin(x-b) \cos(x-a) - \cos(x-b) \sin(x-a)}{\sin(x-a) \sin(x-b)} dx \\
&= \frac{1}{\sin(a-b)} \left[\int \cot(x-a) dx - \int \cot(x-b) dx \right] \\
&= \frac{1}{\sin(a-b)} [\log \sin(x-a) - \log \sin(x-b)] + c \\
&= \frac{1}{\sin(a-b)} \log \left| \frac{\sin(x-a)}{\sin(x-b)} \right| + c
\end{aligned}$$

15 **(b)**

Putting $2+x = t^2$, we get

$$\begin{aligned}
I &= \int \sqrt{\frac{5-x}{2+x}} dx = 2 \int \sqrt{7-t^2} dt \\
\Rightarrow I &= t \sqrt{7-t^2} + 7 \sin^{-1} \frac{t}{\sqrt{7}} + C \\
\Rightarrow I &= \sqrt{x+2} \sqrt{5-x} + 7 \sin^{-1} \frac{\sqrt{x+2}}{7} + C
\end{aligned}$$

16 **(a)**

We have,

$$\begin{aligned}
I &= \int \frac{2x^2+3}{(x^2-1)(x^2+4)} dx = \int \frac{1}{x^2-1} dx + \int \frac{1}{x^2+4} dx \\
\Rightarrow I &= \frac{1}{2} \log \left(\frac{x-1}{x+1} \right) + \frac{1}{2} \tan^{-1} \left(\frac{x}{2} \right) + C \\
\Rightarrow I &= -\frac{1}{2} \log \left(\frac{x+1}{x-1} \right) + \frac{1}{2} \tan^{-1} x + C
\end{aligned}$$

$$\therefore a = -1/2 \text{ and } b = 1/2$$

17 **(c)**

$$\begin{aligned}
\int e^x \frac{(x-1)}{x^2} dx &= \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx \\
&= \frac{e^x}{x} + c
\end{aligned}$$

18 **(d)**

We have,

$$\begin{aligned}
I &= \int_0^{3\alpha} \operatorname{cosec}(x-\alpha) \operatorname{cosec}(x-2\alpha) dx \\
\Rightarrow I &= \frac{1}{\sin \alpha} \int_0^{3\alpha} \frac{\sin \alpha}{\sin(x-\alpha) \sin(x-2\alpha)} dx \\
\Rightarrow I &= \frac{1}{\sin \alpha} \int_0^{3\alpha} \frac{\sin\{(x-\alpha)-(x-2\alpha)\}}{\sin(x-\alpha) \sin(x-2\alpha)} dx
\end{aligned}$$

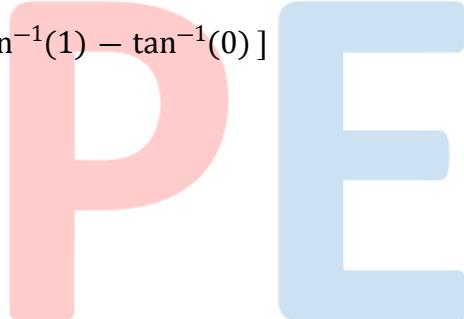
$$\begin{aligned}
 \Rightarrow I &= \frac{1}{\sin \alpha} \int_0^{3\alpha} \cot(x - 2\alpha) - \cot(x - \alpha) dx \\
 \Rightarrow I &= \frac{1}{\sin \alpha} \left[\log \frac{\sin(x - 2\alpha)}{\sin(x - \alpha)} \right]_0^{3\alpha} \\
 \Rightarrow I &= \frac{1}{\sin \alpha} \left[\log \frac{\sin \alpha}{\sin 2\alpha} - \log \frac{\sin 2\alpha}{\sin \alpha} \right] \\
 \Rightarrow I &= \frac{1}{\sin \alpha} \left[\log \left(\frac{\sin \alpha}{2 \sin \alpha \cos \alpha} \right) \right] = \frac{2}{\sin \alpha} \log \left(\frac{1}{2} \sec \alpha \right) \\
 \Rightarrow I &= 2 \operatorname{cosec} \alpha \log \left(\frac{1}{2} \sec \alpha \right)
 \end{aligned}$$

19 (a)

$$\begin{aligned}
 \int_0^3 \frac{3x+1}{x^2+9} dx &= \frac{3}{2} \int_0^3 \frac{2x}{x^2+9} dx + \int_0^3 \frac{1}{x^2+9} dx \\
 &= \frac{3}{2} [\log(x^2+9)]_0^3 + \frac{1}{3} \left[\tan^{-1} \frac{x}{3} \right]_0^3 \\
 &= \frac{3}{2} [\log 18 - \log 9] + \frac{1}{3} [\tan^{-1}(1) - \tan^{-1}(0)] \\
 &= \frac{3}{2} [\log 2] + \frac{\pi}{12} \\
 &= \log(2\sqrt{2}) + \frac{\pi}{12}
 \end{aligned}$$

20 (a)

$$\begin{aligned}
 \text{Let } I &= \int \frac{dx}{\sqrt{(1-x)(x-2)}} \\
 &= \int \frac{dx}{\sqrt{-x^2 + 3x - 2}} = \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2}} \\
 &= \sin^{-1} \left(\frac{\left(x - \frac{3}{2}\right)}{\frac{1}{2}} \right) + c = \sin^{-1}(2x - 3) + c
 \end{aligned}$$



ANSWER-KEY

| | | | | | | | | | | |
|----|----|----|----|----|----|----|----|----|----|----|
| Q. | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| A. | B | A | C | B | D | B | D | A | B | A |
| | | | | | | | | | | |
| Q. | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| A. | D | A | B | A | B | A | C | D | A | A |
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P E