Topic :-Applications of Derivatives

1 **(c)**

From mean value theorem $f'(c) = \frac{f(b) - f(a)}{b - a}$

Given, $a = 0 \Rightarrow f(a) = 0$

and $b = \frac{1}{2} \Rightarrow f(b) = \frac{3}{8}$

Now, $f'(x) = (x - 1)(x - 2) + x(x - 2) + x(x - 1)$

$\therefore f'(c) = (c - 1)(c - 2) + c(c - 2) + c(c - 1)$

$= c^2 - 3c + 2 + c^2 - 2c + c^2 - c$

$\Rightarrow f'(c) = 3c^2 - 6c + 2$

By definition of mean value theorem

$f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow 3c^2 - 6c + 2 = \frac{\left(\frac{3}{8}\right) - 0}{\left(\frac{1}{2}\right) - 0} = \frac{3}{4}$

$\Rightarrow 3c^2 - 6c + \frac{5}{4} = 0$

This is a quadratic equation in c

$\therefore c = \frac{6 \pm \sqrt{36 - 15}}{2 \times 3} = \frac{6 \pm \sqrt{21}}{6} = 1 \pm \frac{\sqrt{21}}{6}$

Since, $'c'$ lies between $\left[0, \frac{1}{2}\right]$

$\therefore c = 1 - \frac{\sqrt{21}}{6} \quad \left(\text{neglecting } c = 1 + \frac{\sqrt{21}}{6}\right)$

2 **(a)**

$\therefore f(x) = \frac{1}{4x^2 + 2x + 1}$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{-(8x + 2)}{(4x^2 + 2x + 1)^2} \quad \text{...(i)}$$

For maxima or minima put $f'(x) = 0$

$$\Rightarrow \ 8x + 2 = 0 \ \Rightarrow \ x = -\frac{1}{4}$$

Again differentiating w.r.t. x of Eq. (i), we get

$$f''(x) = -\frac{\left[\begin{array}{c}(4x^2 + 2x + 1)^2(8) - (8x - 2)2 \times \\ (4x^2 + 2x + 1)(8x + 2)\end{array}\right]}{(4x^2 + 2x + 1)^4}$$

At $x = -\frac{1}{4}, f''\left(-\frac{1}{4}\right) = -$ ve

$f(x)$ is maximum at $x = -\frac{1}{4}$

$\therefore$ maximum value of $f(x)$

$$f\left(-\frac{1}{4}\right)_{max} = \frac{1}{4 \times \frac{1}{16} - 2 \times \frac{1}{4} + 1}$$

$$= \frac{1}{\frac{1}{4} - \frac{2}{4} + 1}$$

$$= \frac{4}{1 - 2 + 4} = \frac{4}{3}$$

4   **(b)**

Let $f(x) = 2x + 3y$

$\therefore \quad f(x) = 2x + \frac{18}{x} \qquad [\because xy = 6, \text{given}]$

$\Rightarrow \quad f'(x) = 2 - \frac{18}{x^2}$

Put $f'(x) = 0$ for maxima or minima

$\Rightarrow 0 = 2 - \frac{18}{x^2} \quad \Rightarrow \quad x = \pm 3$

And $f''(x) = \frac{36}{x^3} \quad \Rightarrow f''(3) = \frac{36}{3^3} > 0$

$\therefore$ At $x = 3, f(x)$ is minimum.

The minimum value is $f(3) = 12$

5   **(c)**

On differentiating given curve w. r. t. x, we get

$$4\,y^3\frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(a,\,a)} = \frac{3a^3}{4a^3} = \frac{3}{4}$$

$\therefore$ Equation of normal at point (a, a) is

$$y - a = -\frac{4}{3}(x - a)$$
$$\Rightarrow 4x + 3y = 7a$$

7   **(b)**

Let $f(x) = x^3 - px + q$

Then $f'(x) = 3x^2 - p$

Put $f'(x) = 0 \Rightarrow x = \sqrt{\frac{p}{3}}, -\sqrt{\frac{p}{3}}$

Now, $f''(x) = 6x$

∴ At $x = \sqrt{\frac{p}{3}}, f''(x) = 6\sqrt{\frac{p}{3}} > 0$, minima

And at $x = -\sqrt{\frac{p}{3}}, f''(x) < 0$, maxima

8   **(a)**

For $x = p, y = ap^2 bp + c$

And for $x = q, y = aq^2 + bq + c$

Slope $= \frac{aq^2 + bq + c - ap^2 - bp - c}{q - p}$

$= a(q + p) + b$

$\frac{dy}{dx} = 2ax + b = a(q + p) + b$

(according to the equation)

∴ $x = \frac{q + p}{2}$

9   **(c)**

Clearly, $f(x)$ is continuous and differentiable on the intervals [0, 3] and (0, 3) respectively for all $n \in N$

Also, $f(0) = f(3) = 0$

It is given that Rolle's theorem for the function $f(x)$ defined on [0, 3] is applicable with $c = \frac{3}{4}$

∴ $f'(c) = 0$

$\Rightarrow 2(c - 3)^n + 2nc(c - 3)^{n-1} = 0$

$[\because f(x) = 2x(x - 3)^n \therefore f'(x) = 2(x - 3)^n + 2nx(x - 3)^{n-1}]$

$\Rightarrow 2(c - 3)^{n-1}(c - 3 + nc) = 0$

$\Rightarrow \frac{3}{4} - 3 + \frac{3}{4}n = 0 \Rightarrow n = 3 \quad [\because c = \frac{3}{4}]$

10   **(a)**

Given,

$y - x = 1$

$\Rightarrow y = x + 1$

$\frac{dy}{dx} = 1$

And $y^2 = x$

$\Rightarrow 2y\frac{dy}{dx} = 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y}$$

$$\therefore \quad 1 = \frac{1}{2y}$$

$$\Rightarrow \quad 2y = 1$$

$$\Rightarrow \quad y = \frac{1}{2}$$

$\therefore$ point on the curve is $\left(\frac{1}{4}, \frac{1}{2}\right)$

$\therefore$ required shortest distance

$$= \left| \frac{\frac{1}{4} - \frac{1}{2} + 1}{\sqrt{2}} \right| = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$$

11 **(d)**

Since, $s^3 \propto v \Rightarrow \frac{ds}{dt} = ks^3 \dots\dots\text{(i)}$

$$\Rightarrow \quad \frac{d^2s}{dt^2} = 3ks^2 \frac{ds}{dt}$$

$$\Rightarrow \quad \frac{d^2s}{dt} = 3k^2s^5 \qquad [from\ Eq.(i)]$$

Hence, acceleration of particle is proportional to s^5.

12 **(a)**

Given, $s = t^3 + 2t^2 + t$

$$\Rightarrow \quad v = \frac{ds}{dt} = 3t^2 + 4t + 1$$

Speed of the particle after 1 s

$$v_{(t=1)} = \left(\frac{ds}{dt}\right)_{(t=1)}$$

$$= 3 \times 1^2 + 4 \times 1 + 1 = 3 + 5 = 8 \text{ cm/s}$$

14 **(b)**

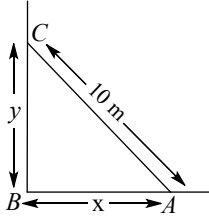Given, $y = a^x \quad \Rightarrow \quad \frac{dy}{dx} = a^x \log a$

Now, length of subtangent at any point $(x_1, \ y_1)$

$$= \frac{y}{dy/dx} = \frac{a^{x_1}}{a^{x_1} \log a} = \frac{1}{\log a}$$

16 **(c)**

Let $AB = x$m, $BC = y$m and $AC = 10$m

$\therefore \quad x^2 + y^2 = 100 \quad \text{...(i)}$

On differentiating w.r.t t,m we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow 2x(3) - 2y(4) = 0$$

$$\Rightarrow x = \frac{4y}{3}$$

On putting this value in Eq. (i), we get

$$\frac{16}{9}y^2 + y^2 = 100$$

$$\Rightarrow y^2 = \frac{100 \times 9}{25} = 36 \Rightarrow y = 6m$$

17   **(c)**

Since $f(x)$, which is of degree 3, has relative minimum/maximum at $x = -1$ and $x = \frac{1}{3}$.

Therefore $x = -1, x = \frac{1}{3}$ are roots of $f'(x) = 0$. Thus, $x + 1$ and $3x - 1$ are factors of $f'(x)$

Consequently, we have

$$f'(x) = \lambda(x + 1)(3x - 1) = \lambda(3x^2 + 2x - 1)$$
$$\Rightarrow f(x) = \lambda(x^3 + x^2 - x) + c$$

Now, $f(-2) = 0$ [Given]

$$\Rightarrow c = 2\lambda \qquad \text{...(i)}$$

We have,

$$\int_{-1}^{1} f(x)dx = \frac{14}{3}$$

$$\Rightarrow \int_{-1}^{1} \left[\lambda(x^3 + x^2 - x) + c\right]dx = \frac{14}{3}$$

$$\Rightarrow \lambda\int_{-1}^{1} x^2 + \int_{-1}^{1} c\, dx = \frac{14}{3} \Rightarrow \frac{2\lambda}{3} + 2c = \frac{14}{3} \Rightarrow \lambda + 3c = 7 \quad \text{...(ii)}$$

Solving (i) and (ii), we get $\lambda = 1, c = 2$

Hence, $f(x) = x^3 + x^2 - x + 2$

18   **(d)**

$$f(x) = \cos x + \frac{1}{2}\cos 2x - \frac{1}{3}\cos 3x$$

$$\Rightarrow f'(x) = -\sin x - \sin 2x + \sin 3x$$

Put $f'(x) = 0$

$\Rightarrow 2\sin\frac{3x}{2}\cos\frac{x}{2} = 2\sin\frac{3x}{2}\cos\frac{3x}{2}$

$\Rightarrow \sin\frac{3x}{2} = 0, \cos\frac{3x}{2} = \cos\frac{x}{2}$

$\Rightarrow x = \frac{2n\pi}{3}, \frac{3x}{2} = 2n\pi \pm \frac{x}{2}$

$\Rightarrow$ At $x = 0, \frac{2\pi}{3}$,

At $x = 0$

$f(x) = 1 + \frac{1}{2} - \frac{1}{3} = \frac{7}{6}$

At $x = \frac{2\pi}{3}$,

$f(x) = -\frac{1}{2} - \frac{1}{4} - \frac{1}{3} = -\frac{13}{12}$

$\therefore$ difference $= \frac{7}{6} + \frac{13}{12} = \frac{27}{12} = \frac{9}{4}$

19   **(a)**

Let $f(x) = e^{x-1} + x - 2$. Then,

$f(1) = e^0 + 1 - 2 = 0$

$\Rightarrow x = 1$ is a real roots of the equation $f(x) = 0$

Let $x = \alpha$ be other real root of $f(x) = 0$ such that $\alpha > 1$

Consider the interval $[1, \alpha]$

Clearly, $f(1) = f(\alpha) = 0$

So, by Rolle's theorem $f'(x) = 0$ has a root in $(1, \alpha)$

But, $f'(x) = e^{x-1} + 1 > 1$ for all x

$\therefore f'(x) \neq 0$ for any $x \in (1, \alpha)$

This is a contradiction

Hence, $f(x) = 0$ has no real root other than 1

20   **(d)**

We have,

$f(x) = \sin^6 x + \cos^6 x$

$\Rightarrow f(x) = (\sin^2 x + \cos^2 x)^3 - 3(\sin^2 x + \cos^2 x)(\sin^2 x \cos^2 x)$

$\Rightarrow f(x) = 1 - 3(\sin^2 x \cos^2 x) = 1 - \frac{3}{4}\sin^2 2x$

$\therefore f(x) \leq 1$, for all x $\left[\because -\frac{3}{4}\sin^2 2x < 0\right]$

and, $f(x) > 1 - \frac{3}{4} = \frac{1}{4}$ for all x $\left[\because -\frac{3}{4}\sin^2 2x \geq -\frac{3}{4}\right]$

ANSWER-KEY

Q.	1	2	3	4	5	6	7	8	9	10
A.	C	A	C	B	C	A	B	A	C	A
Q.	11	12	13	14	15	16	17	18	19	20
A.	D	A	D	B	B	C	C	D	A	D