Class 12^{th}
Relations & Functions

Q.1) Show that the number of equivalence relation in the set {1,2,3} containing (1 , 2) and (2 , 1) is two.

Sol.1) A = {1, 2, 3}
The maximum possible relation (i.e. universal relation) is
R = {(1,1), (2,2), (3,3), (1,2), (1,3), (2,1), (2,3), (3,1), (3,2)}
The smallest equivalence relation R_1 containing (1 , 2) and (2 , 1) is
R_1 = {(1,1), (2,2), (3,3), (1,2), (2,1)}
we are left with four pairs (from universal relation) i.e. (2,3), (3,2), (1,3) and (3,1)
If we add (2,3) to R_1 , then for symmetric by we must add (3,2) and now for transitivity we are forced to add (1,3) and (3,1)
Thus the only relation bigger than R_1 is universal relation i.e R
$\therefore$ The no. of equivalence relations containing (1,2) and (2,1) is two. ans.

Q.2) If $R = \{(x , y) : x^2 + y^2 \le 4 ; x , y \in z\}$ is a relation on z . Write the domain of R.

Sol.2) R = {(0,1), (0,-1), (0,2), (0,-2), (1,1), (1,-1), (-1,0), (-1,1), (-1,-1), (2,0), (-2,0)}
$\therefore$ Domain of R = {0 , 1 , -1 , 2 , -2} ans.
(i.e the first domain of each ordered pairs)

Q.3) Let R =$\{(x, y): |x^2 - y^2| < 1\}$be a relation on set A = {1,2,3,4,5}. Write R as a set of ordered pairs.

Sol.3) A = {1,2,3,4,5}
for $|x^2 - y^2| < 1$: x should be equal to y
$\therefore$ R = {(1,1), (2,2), (3,3), (4,4), (5,5)} ans.

Q.4) R is a relation in Z defined as $(a , b) \in R \Leftrightarrow a^2 + b^2 = 25$. Find the range.

Sol.4) We have, $a^2 + b^2 = 25$ and $a , b \in z$
$\therefore$ R = {(0,5), (0,-5), (3,4), (3,-4), (-3,4), (-3,-4), (4,3), (4,-3), (-4,3), (-4,-3), (5,0), (-5,0)}
$\therefore$ Range = {-5 , 5 , 4 , -4 , 4 , 3 , -3 , 0}
(i.e. second elements of each order pairs) ans.

<u>Topic : Binary Operations</u>

Q.5) $* : R \times R \to R$
(1) a * b = a + b (2) a * b = a – b (3) a * b = ab
Find identity element and inverse in both cases.

Sol.5) (1) a * b = a + b
Identity element

$$\begin{array}{l|l} a * e = a & e * a = a \\ \Rightarrow a + e = a & e + a = a \\ \Rightarrow e = 0 \in R & e = 0 \in R \end{array}$$

$\therefore$ 0 is the identity element
Inverse

$$a * b = e$$
$$\Rightarrow a + b = 0$$
$$\Rightarrow b = -a \in R \qquad \{a \in R \ldots -a \text{ also} \in R\}$$

$\therefore$ –a is the inverse of a i.e a^{-1} = –a

(2) $a * b = a - b$
Identity element

$a * e = a$	$e * a = a$
$a - e = a$	$e - a = a$
$-e = 0$	$e = 2a$
$e = 0 \in R$	but 'e' cannot be a variable as when a changes e also change but should be same for all $a \in R$

$\therefore$ Identity element does not exist
hence inverse does not exist

(3) $a * b = ab$
Identity element

$a * e = a$	$e * a = a$
$a e = a$	$e a = a$
$e = 1 \in R$	$e = 1 \in R$

$\therefore$ 1 is the identity element
Inverse
$a * b = e$
$ab = 1$
b $= \frac{1}{a} \in R; a \neq 0$
$\therefore$ all elements of R are invertible except '0' and $a^{-1} = \frac{1}{a}; a \neq 0$ ans.

Q.6) Let * be a binary operation on R (real no's)
$* : R \times R \to R$
$a * b = a + b + ab$
(.) Check whether * is binary operation or not
(.) Check the commutativity and Associativity
(.) Find identity element and inverse.

Sol.6) We have,
$a * b = a + b + ab$
since * carries each pair (a , b) in $R \times R$to a unique element $a + b + ab$ in R
$\therefore$ * is a binary operation on R
Alternate : since$(a, b) \in R \times R$and addition and multiplication of real no.s is also a real no.
$a + b + ab \in R$
$\therefore$ * is a binary operation on R
Commutative :
let $a , b \in R$
$a * b = a + b + ab$
$b * a = b + a + ba$
$= a + b + ab$ ....{$\because$ addition and multiplication are itself commutative}
$= a * b$
$\therefore$ $b * a = a * b$ for all $a , b \in R$
$\therefore$ * is commutative on R
Associative:

let $a, b, c \in R$

$(a * b) * c = (a + b + ab) * c$

$= a + b + ab + c\,(a + b + ab)c$

$= a + b + ab + c + ac + bc + abc$

$= a + b + c + ab + bc + ac + abc$

Now $a * (b * c) = a * (b + c + bc)$

$= a + b + c + bc + a\,(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

$= a + b + c + ab + bc + ac + abc$

clearly $(a * b) * c = a * (b * c)$ for all $a, b, c \in R$

$\therefore$ * is Associative on R.

Identity element :

let e be the identity element in R

$a * e = a$ and $e * a = a$ for all $a \in R$

$\Rightarrow a + e + ae = a$	$e + a + ea = a$
$\Rightarrow e(1 + a) = 0$	$e\,(1 + a) = 0$
$\Rightarrow e = 0 \in R$	$e = 0 \in R$

$\therefore$ 0 is the identity element

Inverse :

$a * b = e$

$a + b + ab = 0$

$b(1 + a) = -a$

b = $\frac{-a}{1+a} \in R$ {except a = –1}

$\therefore$ –1 is not the invertible element

(.) all elements of R are invertible except –1

(.) and $a^{-1} = \frac{-a}{1+a}; a \neq -1$

(.) e.g. inverse of $2 = \frac{-2}{1+2} = \frac{-2}{3}$ ans.

Q.7) Let * be a binary operation on Z (integers) $a * b = a + ab$
Check the commutative , Associativity , identify element and inverse (if it exists).

Sol.7) We have

$a * b = a + ab$ where $a, b \in z$

Commutative :

let $a, b \in z$, then

$a * b = a + ab$

$b * a = b + ba$

$= b + ab$

$b * a \neq a * b$

e.g. $(1 * 2) = 1 + (1)(2) = 1 + 2 = 3$

$(2 * 1) = 2 + 2(1) = 2 + 2 = 4$

clearly $1 * 2 \neq 2 * 1$

$\therefore$ * is not commutative on Z

Associative :

let $a, b, c \in Z$ then

$(a * b) * c = (a + ab) * c$

$= a + ab + (a + ab)\, c$
$= a + ab + ac + abc$
$a * (b * c) = a * (b + bc)$
$= a + a(b + bc)$
$= a + ab + abc$
$\neq (a * b) * c$
e.g. $(1 * 2) * 3 = (1 + 2) * 3$
$= 3 * 3$
$= 3 + 3(3) = 12$
$1 * (2 * 3) = 1 * (2 + 6) = 1 * 8$
$= 1 + 1(8) = 9$
Clearly * is not Associative on Z
Identity element:
let e be the identity element ? z , then

$a * e = a$	$e * a = a$
$a + ae = a$	$e + ea = a$
$ae = 0$	$e(1 + a) = a$
$e = 0 \in z$	$e = \frac{a}{1+a}$

as a changes e also changes , but e must be constant for all value of a
$\therefore$ identity element does not exist and hence inverse not possible not possible. ans.

Q.8) Let * be a binary operation on N given by a * b = LCM of a & b
(1) Find $5 * 7$, $20 * 16$
(2) Is * commutative ?
(3) If * Associative ?
(4) Find the identity element in N.
(5) which elements of N are invertible ?

Sol.8) We have
$a * b$ = LCM of a & b ; $a, b \in N$
(1) 5 * 7 = LCM of 5 and 7 = 35
20 * 16 = LCM of 20 and 16 = 80
(2) Commutative :
let $a , b \in N$
a * b = LCM of a and b
b * a = LCM of b and a
= LCM of a and b
= a * b
$\therefore$ b * a = a * b for all $a, b \in N$
$\therefore$ * is commutative on N
(3) Associative :
let $a , b , c \in N$
$(a * b) * c = (LCM\ of\ a\ and\ b) * c$
$= LCM\ of$ [$(LCM\ of\ a$ and $b)$ $and\ c$]
$= LCM\ of\ a , b\ and\ c$
$a * (b * c) = a * (LCM\ of\ b\ and\ c)$
= LCM of [$a\ and\ (LCM\ of\ b\ and\ c)$]
= LCM of $a , b\ and\ c$

clearly $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$
$\therefore$ * is Associative on N
(4) Identity element
let e be an identity element $\in N$
$a * e = a$
$\Rightarrow$ LCM of a and $e = a$ $e * a = a$
$\Rightarrow$ LCM of a & $1 = a$ LCM of e and $a = a$
$\Rightarrow e = 1 \in N$ LCM of 1 and $a = a$
$\Rightarrow e = 1 \in N$
$\therefore$ 1 is the identity element for all $a \in N$
Inverse
$a * b = e$
$\Rightarrow$ LCM of a and $b = 1$
this is possible only when $a = 1$ & $b \neq 1$
$\therefore$ 1 is the only invertible element and 1 is its inverse ans.

Q.9) Let R be a of real no.s and $A = R \times R$ is a binary operation on A given by $(a, b) * (c, d) = (ac, bd)$ for all (a, b) $(c, d) \in A$
(1) Show that * is Commutative
(2) Show that * is Associative
(3) Find the identity element
(4) Find invertible elements and their inverse.

Sol.9) We have,
$(a, b) * (c, d) = (ac, bd)$
Commutative :
let (a, b) & $(c, d) \in A, then$
$(a, b) * (c, d) = (ac, bd)$
$(c, d) * (a, b) = (ca, db)$
$= (ac, bd)$
$= (a, b) * (c, d)$
$\therefore$ * is commutative on A
Associative :
let $(a, b), (c, d)$ & $(e, f) \in A$
$[(a, b)\ (c, d)] * (e, f)$
$= (ac, bd)\ (e, f)$
$= (ace, bdf)$
$(a, b) * [(c, d) * (e, f)]$
$= (a, b) * (ce, df)$
$= (ace, bdf)$
clearly $((a, b) * (c, d)) * (e, f) = (a, b) * ((c, a) * (e, f))$
$\therefore$ * is Associative on A
Identity element
let (x, y) be the identity element

$(a, b) * (x, y) = (a, b)$	$(x, y) * (a, b) = (a, b)$
$\Rightarrow (ax, by) = (a, b)$	$\Rightarrow (xa, yb) = (a, b)$
$\Rightarrow ax = a$ & $by = b$	$\Rightarrow xa = a$ & $yb = b$
$\Rightarrow x = 1$ and $y = 1$	$\Rightarrow x = 1$ & $y = 1$

$\therefore$ (1, 1) is the identity element

Inverse

$(a\,,b)\ *\ (c\,,d)\ =\ (x\,,y)$

$\Rightarrow\ (ac\,,bd)\ =\ (1\,,1)$

$\Rightarrow\ ac\ =\ 1\ and\ bd\ =\ 1$

$\Rightarrow\ c\ = \dfrac{1}{a}\ and\ d\ = \dfrac{1}{b}$

$\therefore\ (c\,,d) = \left(\frac{1}{a},\frac{1}{b}\right) \in R$ except $(0\,,b)\ =\ (0\,,0)$

(.) all elements of A are invertible except (0 , 0)

(.) inverse of (0 , b) is $\left(\frac{1}{a},\frac{1}{b}\right)$; (a , b) ≠ (0 , 0) ans.

Q.10) X is a non-empty set and * is a binary operation $*\!:\ p(x)\ * P(x)\ \rightarrow\ P(x)$ given by $A\ \times\ B = A\ \cap\ B$

(.) Show * is Commutative

(.) Show * is Associative

(.) Find the Identity element

(.) Find the Invertible elements in $P(x)$ and their inverse.

Sol.10) We have, $A\ \times\ B\ =\ A\ \cap\ B$

Commutative :

let $A\,,B\ \in\ P(x)$

$A\ *\ B\ =\ A\ \cap\ B$

$B\ *\ A\ =\ B\ \cap\ A$

$=\ A\ \cap\ B$ ......{$\because$ ∩ is its of commutative}

clearly $A\ *\ B\ =\ B\ *\ A$ for all $A,B\ \in\ P(x)$

$\therefore$ * is commutative on P(x)

Associative :

let $A\,,B\,,C\ \in\ P(x)$

$(A\ *\ B) *\ C = (A\ \cap\ B) *\ C$

$=\ (A\ \cap\ B)\ \cap\ C$

$A\ *\ (B\ *\ C) =\ A\ *\ (B\ \cap\ C)$

$=\ A\ \cap\ (B\ \cap\ C)$ $\{\cap\ is\ itself\ Associative\}$

clearly $(A\ *\ B)\ *\ C\ =\ A\ *\ (B\ *\ C)$ for all $A,B,C\ \in\ P(x)$

$\therefore$ * is Associative on $P(x)$

Identity element

let E is an identity element then

$A\ *\ E\ =\ A$ $\qquad$ $E\ *\ A\ =\ A$

$\Rightarrow\ A\ \cap\ E\ =\ A$ $\qquad$ $\Rightarrow\ E\ \cap\ A\ =\ A$

$\Rightarrow\ E\ =\ X\ \in\ (Px)$ $\qquad$ $\Rightarrow\ E\ =\ X\ \in\ P(x)$

{reason : X is the largest subset in P(x)}

$\therefore$ X is the identity element

Inverse :

$A\ *\ B\ =\ E$

$\Rightarrow\ A\ \cap\ B\ =\ X$

this is possible only when $A\ =\ B\ =\ X$

since $X\ \cap\ X\ =\ A$

$\therefore$ X is only the invertible element in $P(x)$ and X is its inverse ans.