Integration (Indefinite Integrals)

→	**Type: (.)** $I=\int e^{ax}\sin(bx+c)dx$ $I=\int e^{ax}\cos(bx+c)dx$ **I repeats of the two types by Parts**
Q.1)	(a)$I=\int e^{2x}.\cos(3x)dx$ (b)$I=\int e^{ax}.\sin(dx+c)dx$
Sol.1)	(a)$I=\int e^{2x}.\cos(3x)dx$ $=\cos(3x).\frac{e^{2x}}{2}-\int(-3\sin(3x)).\frac{e^{2x}}{2}dx$ $=\frac{e^{2x}}{2}.\cos(3x)+\frac{3}{2}\int\sin(3x).e^{2x}dx$ $=\frac{e^{2x}}{2}.\cos(3x)+\frac{3}{2}\left[\sin(3x).\frac{e^{2x}}{2}-\int 3\cos(3x).\frac{e^{2x}}{2}dx\right]$ $I=\frac{e^{2x}}{2}.\cos(3x)+\frac{3}{2}\left[\frac{e^{2x}}{2}.\sin 3x-\frac{3}{2}I\right]$ $I=\frac{e^{2x}}{2}.\cos(3x)+\frac{3}{4}e^{2x}.\sin(3x)-\frac{9}{4}I$ $I+\frac{9}{4}I=\frac{e^{2x}}{4}[2\cos(3x)+3\sin(3x)]$ $\frac{13I}{4}=\frac{e^{2x}}{4}[2\cos(3x)+3\sin(3x)]+c$ $\therefore I=\frac{e^{2x}}{13}[2\cos(3x)+3\sin(3x)]+c$ ans. (b)$I=\int e^{ax}.\sin(bx+c)dx$ $=\sin(bx+c).\frac{e^{ax}}{a}-\int b\cos(bx+c).\frac{e^{ax}}{a}dx$ $=\frac{e^{ax}}{a}\sin(bx+c)-\frac{b}{a}\int e^{ax}.\cos(bx+c)dx$ $=\frac{e^{ax}}{a}.\sin(bx+c)-\frac{b}{a}\left[\cos(bx+c).\frac{e^{ax}}{a}-\int(-b\sin(bx+c)).\frac{e^{ax}}{a}dx\right]$ $=\frac{e^{ax}}{a}.\sin(bx+c)-\frac{b}{a}\left[\frac{e^{ax}}{a}.\cos(bx+c)+\frac{b}{a}\int e^{ax}.\sin(bx+c)dx\right]$ $=\frac{e^{ax}}{a}.\sin(bx+c)-\frac{b}{a}\left[\frac{e^{ax}}{a}\cos(bx+c)+\frac{b}{a}I\right]$ $I=\frac{e^{ax}}{a}.\sin(bx+c)-\frac{b}{a^2}e^{ax}.\cos(bx+c)-\frac{b^2}{a^2}I$ $I+\frac{b^2}{a^2}I=\frac{e^{ax}}{a^2}[a\sin(bx+c)-b\cos(bx+c)]+c$ $I\left(\frac{a^2+b^2}{a^2}\right)=\frac{e^{ax}}{a^2}[a\sin(bx+c)-b\cos(bx+c)]+c$ $\therefore I=\frac{e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]+c$ ans.
Q.2)	$I=\int e^{x}.\cos^2 x\,dx$

Sol.2)	$I = \int e^x . \cos^2 x dx$ $= \int e^x . \left\{\frac{1+\cos(2x)}{2}\right\} dx$ $= \frac{1}{2}\int e^x + e^x . \cos(2x) dx$ $I = \frac{1}{2}\int e^x dx + \frac{1}{2}\int e^x . \cos(2x) dx$ $I = \frac{1}{2}\int e^x dx + \frac{1}{2} I_1$ where $I = \int e^x . \cos(2x) dx$ $= \cos(2x). e^x - \int -2\sin(2x). e^x dx$ $= e^x . \cos(2x) + 2\int e^x . \sin(2x) dx$ $= e^x . \cos(2x) + 2[e^x . \sin(2x) - 2\int \cos(2x). e^x dx]$ $I_1 = e^x \cos(2x) + 2e^x \sin(2x) - 4I_1$ $5I_1 = e^x[\cos(2x) + 2\sin(2x)]$ $I_1 = \frac{e^x}{5}[\cos(2x) + 2\sin(2x)] + c$ $\therefore I = \frac{1}{2}e^x + \frac{1}{2}\left[\frac{e^x}{5} . (\cos(2x) + 2\sin(2x))\right] + c$ ans.
→	**Type**: $I = \int e^x (f(x) + f'(x)) dx$ $I = \int e^x . f(x) dx + \int e^x . f'(x) dx$ $= f(x). e^x - \int f'(x). e^x dx + \int e^x . f'(x) dx$ $I = e^x . f(x) + c$
Q.3)	(a) $I = \int e^x \left(\frac{2+\sin(2x)}{1+\cos(2x)}\right) dx$ (b) $I = \int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$ (c) $I = \int e^{2x} \left(\frac{1+\sin(2x)}{1+\cos(2x)}\right) dx$
Sol.3)	(a) $I = \int e^x \left(\frac{2+\sin(2x)}{1+\cos(2x)}\right) dx$ $= \int e^x \left[\frac{2+2\sin x . \cos x}{2\cos^2 x}\right] dx$ $= \int e^x \left[\frac{2}{2\cos^2 x} + \frac{2\sin x . \cos x}{2\cos^2 x}\right] dx$ $= \int e^x \underset{f'(x) f(x)}{(sec^2 x + \tan x)} dx$ $= \int e^x . \tan x \, dx + \int e^x sec^2 x \, dx$ $= \tan x . e^x - \int sec^2 x . e^x dx + \int e^x sec^2 x \, dx$ $= e^x . \tan x + c$ ans. (b) $I = \int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$ $= \int e^x \left[\frac{1-2\sin\frac{x}{2} . \cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right] dx$

	$= \int e^x \left[\frac{1}{2\sin^2\frac{x}{2}} - \frac{2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}\right] dx$ $= \int e^x \left[\underset{f'(x)}{\frac{1}{2}cosec^2\left(\frac{x}{2}\right)} - \underset{f(x)}{cot\left(\frac{x}{2}\right)}\right] dx$ $= -\int e^x.\cot\frac{x}{2} dx + \frac{1}{2}\int e^x.cosec^2\frac{x}{2} dx$ $= -\left[\cot\frac{x}{2}.e^x - \int -\frac{1}{2}cosec^2\frac{x}{2}.e^x dx\right] + \frac{1}{2}\int e^x.cosec^2\frac{x}{2} dx$ $= -e^x\cot\frac{x}{2} - \frac{1}{2}\int e^x.cosec^2\frac{x}{2} dx + \frac{1}{2}\int e^x.cosec^2\frac{x}{2} dx$ $= I = -e^x.\cot\frac{x}{3} + c$ ans. (c)$I = \int e^{2x}\left(\frac{1+\sin(2x)}{1+\cos(2x)}\right) dx$ $= \int e^{2x}\left(\frac{1+2\sin x\cos x}{2\cos^2 x}\right) dx$ $= \int e^{2x}.\left(\frac{1}{2}sec^2 x + \tan x\right) dx$ $= \int e^{2x}.\tan x\, dx + \frac{1}{2}\int e^{2x}.sec^2 x\, dx$ $= \tan x.\frac{e^{2x}}{2} - \int sec^2 x.\frac{e^{2x}}{2} dx + \frac{1}{2}\int e^{2x}.sec^2 x\, dx$ $I = \frac{1}{2}e^{2x}.\tan x + c$ ans.
Q.4)	(a)$I = \int e^x - \frac{x}{(x+1)^2} dx$ (b)$I = \int e^x\left(\frac{x-4}{(x-2)^3}\right) dx$
Sol.4)	(a)$I = \int e^x - \frac{x}{(x+1)^2} dx$ $= \int e^x\left[\frac{x+1-1}{(x+1)^2}\right] dx$ $= \int e^x\left[\underset{f(x)}{\frac{1}{x+1}} - \underset{f'(x)}{\frac{1}{(x+1)^2}}\right] dx$ $= \int e^x.\frac{1}{x+1} dx - \int e^x.\frac{1}{(x+1)^2} dx$ $= \frac{1}{x+1}.e^x + \int \frac{1}{(x+1)^2}.e^x dx - \int \frac{1}{(x+1)^2}.e^x dx$ $I = e^x.\frac{1}{x+1} + c$ ans. (b)$I = \int e^x\left(\frac{x-4}{(x-2)^3}\right) dx$ $= \int e^x\left(\frac{x-4}{(x-2)^3}\right) dx$ $= \int e^x\left[\underset{f(x)}{\frac{1}{(x-2)^2}} - \underset{f'(x)}{\frac{2}{(x-2)^3}}\right] dx$ Proceed Yourself $e^x.\frac{1}{(x-2)^2} + c$ ans.

Q.5)	$I = \int e^x . \frac{(x^2+1)}{(x+1)^2} dx$
Sol.5)	$I = \int e^x . \frac{(x^2+1)}{(x+1)^2} dx$ $= \int e^x . \left[\frac{x^2+1+2x-2x}{(x+1)^2}\right] dx$ $= \int e^x \left(\frac{x^2+1+2x}{(x+1)^2} - \frac{2x}{(x+1)^2}\right) dx$ $= \int e^x \left(1 - \frac{2x}{(x+1)^2}\right) dx$ $= \int e^x dx - 2\int e^x . \frac{x}{(x+1)^2} dx$ $= e^x - 2\int e^x . \left[\frac{x+1-1}{(x+1)^2}\right] dx$ $= e^x - 2\int e^x \left(\frac{1}{x+1} - \frac{1}{(x+1)^2}\right) dx$ $= e^x - 2\left[\int e^x . \frac{1}{x+1} dx - \int e^x . \frac{1}{(x+1)^2} dx\right]$ $= e^x - 2\left[\frac{1}{(x+1)} . e^x + \int \frac{1}{(x+1)^2} . e^x dx - \int e^x . \frac{1}{(x+1)^2} dx\right]$ $= e^x - 2 . \frac{e^x}{x+1} + c$ $= e^x \left(1 - \frac{2}{(x+1)}\right) + c$ $= e^x \left(\frac{x-1}{x+1}\right) + c$ ans.
Q6)	(a)$I = \int \frac{\log x}{(\log x+1)^2} dx$ (b)$I = \int \log(\log x) + \frac{1}{(\log x)^2} dx$
Sol.6)	(a)$I = \int \frac{\log x}{(\log x+1)^2} dx$ put $\log x = t$ $x = e^t$ $dx = e^t dt$ $\therefore I = \int \frac{t}{(t+1)^2} . e^t dt$ $= \int e^t \left[\frac{t+1-1}{(t+1)^2}\right] dt$ $= \int e^t \left[\frac{1}{t+1} - \frac{1}{(t+1)^2}\right] dt$ $= e^t . \frac{1}{t+1} + c$ replacing t $= x . \frac{1}{\log x+1} + c$ ans. (b)$I = \int \log(\log x) + \frac{1}{(\log x)^2} dx$ put $\log x = t$

	$x = e^t$ $dx = e^t\,dt$ $\therefore I = \int \left(\log t + \frac{1}{t^2}\right).e^t dt$ adjustment $= \int e^t \left[\log t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2}\right] dt$ $= \int e^t \left(\log t + \frac{1}{t}\right) dt - \int e^t \left(\frac{1}{t} - \frac{1}{t^2}\right) dt$ $= \left[\int e^t \log t\, dt + \int e^t.\frac{1}{t} dt\right] - \left[\int e^t.\frac{1}{t} dt - \int e^t.\frac{1}{t^2} dt\right]$ $= \left[\log t\,.e^t - \int \frac{1}{t}.e^t dt + \int e^t.\frac{1}{t} dt\right] - \left[\frac{1}{t}.e^t + \int \frac{1}{t^2}.e^t dt - \int e^t.\frac{1}{t^2} dt\right]$ $= \log t\,.e^t - \frac{1}{t}.e^t + c$ $= e^t \left(\log t - \frac{1}{t}\right) + c$ $I = x\left(\log(\log x) - \frac{1}{\log x}\right) + c$ ans.
Q.7).	(a)$I = \int e^{\frac{-x}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x} dx$ (b)$I = \int e^{2\text{x}}(-\sin x + 2\cos x) dx$
Sol.7)	(a)$I = \int e^{\frac{-x}{2}} \frac{\sqrt{1-\sin x}}{1+\cos x} dx$ $= \int e^{\frac{-x}{2}} \frac{\sqrt{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}-2\sin\frac{x}{2}.\cos\frac{x}{2}}}{2\cos^2\left(\frac{x}{2}\right)} dx$ $= \int e^{\frac{-x}{2}} \frac{\sqrt{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)^2}}{2\cos^2\frac{x}{2}}$ $= \int e^{\frac{-x}{2}} \frac{\left(\sin\frac{x}{2}-\cos\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} dt$ $= \int e^{\frac{-x}{2}} \left[\frac{1}{2}\tan\frac{x}{2}.sec\frac{x}{2} - \frac{1}{2}sec\left(\frac{x}{2}\right)\right] dx$ $= \frac{-1}{2}\int e^{\frac{-x}{2}} sec\left(\frac{x}{2}\right) dx + \frac{1}{2}\int e^{\frac{-x}{2}} sec\frac{x}{2}.\tan\frac{x}{2} dx$ $= \frac{-1}{2}\left[sec\frac{x}{2}.e^{\frac{-x}{2}}(-2) - \int sec\left(\frac{x}{2}\right).\tan\frac{x}{2}.\left(\frac{1}{2}\right).e^{\frac{-x}{2}}(-2)dx\right] + \frac{1}{2}\int e^{\frac{-x}{2}}.sec\frac{x}{2}\tan\frac{x}{2} dx$ $= e^{\frac{-x}{2}}.sec\frac{x}{2} - \frac{1}{2}\int e^{\frac{-x}{2}}.sec\frac{x}{2}\tan\frac{x}{2} dx + \frac{1}{2}\int e^{\frac{-x}{2}} sec\frac{x}{2}\tan\frac{x}{2} dx$ $= e^{\frac{-x}{2}} sec\left(\frac{x}{2}\right) + c$ ans.
→	**Type**: $\int \sqrt{Quadratic}$ and $\int Linear\sqrt{Quadratic}$ **Perfect Square. Use Long Formula**
Q.8)	(a)$I = \int \sqrt{(x-3)(5-x)} dx$ (b)$I = \int \sqrt{2\text{x}^2 + 3\text{x} + 4}\, dx$ (c)$I = \int \sqrt{3 - 2\text{x} - 2\text{x}^2} dx$

Sol.8)	(a)$I = \int \sqrt{(x-3)(5-x)}dx$ $= \int \sqrt{5x - x^2 - 15 + 3x}dx$ $= \int \sqrt{-x^2 + 8x - 15}dx$ $= \int \sqrt{-[x^2 - 8x - 15]}dx$ $= \int \sqrt{-[x^2 - 8x + 15]}dx$ $= \int \sqrt{-[(x-4)^2 + 6 + 15]}dx$ $= \int \sqrt{-[(x-4)^2 - 1]}dx$ $= \int \sqrt{1^2 - (x-4)^2}dx$ $= \frac{(x-4)}{2}\sqrt{1-(x-4)^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-4}{1}\right) + c$ $= \frac{(x-4)}{2}\sqrt{(x-3)(5-x)} + \frac{1}{2}\sin^{-1}(x-4) + c$ ans.
Q.9)	$I = \int \cos x\sqrt{4 - \sin^2 x}dx$
Sol.9)	$I = \int \cos x\sqrt{4 - \sin^2 x}\, dx$ put $sin\ x = t$ $\therefore\ cos\ x\ dx = dt$ $I = \int \sqrt{4 - t^2}dt$ $= \frac{1}{2}\sqrt{4 - t^2} + 2\sin^{-1}\left(\frac{t}{2}\right) + c$ $= \frac{\sin x}{x}\sqrt{4 - \sin^2 x} + 2\sin^{-1}\left(\frac{\sin x}{2}\right) + c$ ans.
Q.10)	(a)$I = \int (3x-2)\sqrt{x^2 + x + 1}dx$ (b) $I = \int (4x+1)\sqrt{x^2 - x - 2}dx$
Sol.10)	(a)$I = \int (3x-2)\sqrt{x^2 + x + 1}dx$ (take $2x + 1$) $= 3\int \left(x - \frac{2}{3}\right)\sqrt{x^2 + x + 1}dx$ $= \frac{3}{2}\int \left(2x - \frac{4}{3}\right)\sqrt{x^2 + x + 1}dx$ $= \frac{3}{2}\int \left(2x - \frac{4}{3} + 1 - 1\right)\sqrt{x^2 + x + 1}dx$ $= \int \sqrt{5x - x^2 - 15 + 3x}dx$ $= \frac{3}{2}\int (2x+1)\sqrt{x^2 + x + 1}dx - \frac{7}{2}\int \sqrt{x^2 + x + 1}dx$ put $x^2 + x + 1 = t$ in I $(2x+1)dx = dt$ $= \frac{3}{2}\int \sqrt{t}dt - \frac{7}{2}\int \sqrt{\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} + 1dx}$ $= \frac{3}{2} \times \frac{2}{3}(\sqrt{t})^{\frac{3}{2}} - \frac{7}{2}\int \sqrt{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\, dx$

$= (\sqrt{t})^{\frac{3}{2}} - \frac{7}{2}\left[\frac{(x+\frac{1}{2})}{2}\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{3}{8}\log\left|\left(x+\frac{1}{2}\right) + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right|\right]$

$= (\sqrt{x^2+x+1})^{\frac{3}{2}} - \frac{7}{2}\left[\frac{(2x+1)}{4}\sqrt{x^2+x+1} + \frac{3}{8}\log\left|\frac{2x+1}{2} + \sqrt{x^2+x+1}\right|\right]$

(b) $I = \int (4x+1)\sqrt{x^2-x-2}\,dx$

$= 2\int \left(2x+\frac{1}{2}\right)\sqrt{x^2-x-2}\,dx$

$= 2\int \left(2x+\frac{1}{2}-1+1\right)\sqrt{x^2-x-2}\,dx$

$= 2\int \left(2x-1+\frac{3}{2}\right)\sqrt{x^2-x-2}\,dx$

$= 2\int (2x-1)\sqrt{x^2-x-2}\,dx + 3\int \sqrt{x^2-x-2}\,dx$

put $x^2 - x - 2 = t$

$(2x-1)dx = dt$

$\therefore I = 2\int dt + 3\int \sqrt{\left(x-\frac{1}{4}\right)^2 - \frac{1}{4} - 2}\,dx$

$= 2 \times \frac{2}{3}(t)^{\frac{3}{2}} + 3\int \sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\,dx$

$= \frac{4}{3}(t)^{\frac{3}{2}} + 3\left[\frac{x-\frac{1}{2}}{2}\sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2} - \frac{9}{8}\log\left|\left(x-\frac{1}{2}\right) + \sqrt{\left(x-\frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}\right|\right]$

$= \frac{4}{3}(x^2-x-2)^{\frac{3}{2}} + 3\left[\frac{2x-1}{4}\sqrt{x^2-x-2} - \frac{9}{8}\log\left|\frac{2x-1}{2} + \sqrt{x^2-x-2}\right|\right]$ ans.