Integration (Indefinite Integrals)

Q.1)	$I = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$
Sol.1)	$I = \int \sqrt{\dfrac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$ put $x = t^2$ $dx = 2\,t\,dt$ $\therefore = 2\int \sqrt{\frac{1-t}{1+t}}.\,tdt$ rationalize $= 2\int \sqrt{\frac{1-t}{1+t} \times \frac{1-t}{1-t}}.\,tdt$ $= 2\int \frac{(1-t)}{\sqrt{1-t^2}}.\,tdt$ $= 2\int \frac{t-t^2}{\sqrt{1-t^2}}dt$ $= 2\int \frac{t}{\sqrt{1-t^2}}dt - 2\int t^2\sqrt{1-t^2}dt$ put $1-t^2 = z$ in (I) $-2t\,dt = dz$ $t\,dt = -\frac{dz}{2}$ $\therefore I = -\frac{2}{2}\int \frac{dz}{\sqrt{z}} + 2\int -\frac{t^2}{\sqrt{1-t^2}}dt$ $= -2\sqrt{z} + 2\int \frac{1-t^2-1}{\sqrt{1-t^2}}dt$ $= -2\sqrt{1-t^2} + 2\int \sqrt{1-t^2} - \frac{1}{\sqrt{1-t^2}}dt$ $= -2\sqrt{1-t^2} + 2\left[\frac{t}{2}\sqrt{1-t^2} + \frac{1}{2}\sin^{-1}(t) - \sin^{-1}(t)\right] + c$ $= -2\sqrt{1-t^2} + 2\left[\frac{t}{2}\sqrt{1-t^2} - \frac{1}{2}\sin^{-1}t\right] + c$ $= -2\sqrt{1-t^2} + t\sqrt{1-t^2} - \sin^{-1}t + c$ replacing t by $\sqrt{x}$ $= I = -2\sqrt{1-x} + \sqrt{x}\sqrt{1-x} - \sin^{-1}\sqrt{x} + c$ ans.
→	**Partial Fraction (Total : Types)** **Type : 1 all are linear factors (ax + b)**
Q.2)	(a) $I = \int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$ (b) $I = \int \frac{x^3}{(x-1)(x-2)}dx$ (c) $I = \int \frac{x^2}{(x-1)(x-2)(x-3)}dx$ (d) $I = \int \frac{3+4x-x^2}{(x+2)(x-1)}dx$
Sol.2)	(a) $I = \int \frac{2x-1}{(x-1)(x+2)(x-3)}dx$ let $\frac{2x-1}{(x-1)(x+2)(x-3)} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{x-3}$ $\Rightarrow 2x-1 = A(x+2)(x-3) + B(x-1)(x-3) + C(x-1)(x+2)$

	$\Rightarrow 2x - 1 = A(x^2 - x - 6) + B(x^2 - 4x + 3) + C(x^2 + x - 2)$ Comp. the coefficients of x^2 , x and constant term $0 = A + B + C \quad \Rightarrow \quad C = -A - B$ $2 = -A - 4B + C \quad \Rightarrow \quad 2 = -2A - 5B$ $-1 = -6A + 3B - 2C \Rightarrow -1 = -4A - B$ solving these two equation we get $A = \frac{-1}{6}, B = \frac{-1}{3}$ and $C = \frac{1}{2}$ $\therefore I = \int \frac{-1}{6(x-1)} - \frac{1}{3(x+2)} + \frac{1}{2(x-3)} dx$ $= \frac{-1}{6}\log \mid x - 1 \mid - \frac{1}{3}\log \mid x + 2 \mid + \frac{1}{2}\log \mid x - 3 \mid + c$ ans. (b) $I = \int \frac{x^3}{(x-1)(x-2)} dx$ Since degree of N[r] > degree of D[r] we have to divide $\therefore I = \int (x + 3) + \frac{7x-6}{(x-1)(x-2)} dx$ $= \frac{x^2}{2} + 3x + \int \frac{7x-6}{(x-1)(x-2)} dx$ let $\frac{7x-6}{(x-1)(x-2)} = \frac{A}{x-1} + Bx - 2$ $\Rightarrow 7x - 6 = A(x-2) + B(x-1)$ Comp. the coefficient of x and constant term $7 = a + b$ $-6 = -2A - B$ $1 = -A$ $\therefore B = 8$ $A = -1 \ and \ B = 8$ $\therefore I = \frac{x^2}{2} + 3x + \int \frac{-1}{x-1} + \frac{8}{x-2} dx$ $I = \frac{x^2}{2} + 3x - \log \mid x - 1 \mid + 8\log \mid x - 2 \mid + c$ ans.
Q.3)	$I = \int \frac{x}{(x^2+1)(x^2+2)} dx$
Sol.3)	$I = \int \frac{x}{(x^2+1)(x^2+2)} dx$ put $x^2 = t$ $x\,dx = \frac{t}{2}$ $\therefore I = \frac{1}{2}\int \frac{dt}{(t+1)(t+2)}$ let $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$ Proceed Yourself $\frac{1}{2}[\log \mid x^2 + 1 \mid - \log \mid x^2 + 2 \mid] + c$ ans.
Q.4)	(a) $I = \int \frac{1}{\sin x - \sin(2x)} dx$

Sol.4)	(a)$I = \int \frac{1}{\sin x - \sin(2x)} dx$ $= \int \frac{1}{\sin x 2 \sin x \cos x} dx$ $= \int \frac{1}{\sin x(1-2\cos x)} dx$ multiply and divide by $sin\ x$ $= \int \frac{\sin x}{\sin^2 x(1-2\cos x)} dx$ $= \int \frac{\sin x}{(1-\sin^2 x)(1-\cos x)} dx$ $= \int \frac{\sin x}{(1-\cos^2 x)(1-2\cos x)} dx$ $= \int \frac{\sin x}{(1-\cos x)(1+\cos x)(1-2\cos x)} dx$ put $cos\ x = t$ $\therefore\ sin\ x\ dx = -dt$ $\therefore I = -\int \frac{dt}{(1-t)(1+t)(1-2t)}$ let $\frac{1}{(1-t)(1+t)(1-2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1-2t}$ $\Rightarrow\ 1 = A(1+t)(1-2t) + B(1-t)(1-2t) + C(1-t)(1+t)$ $\Rightarrow\ 1 = A(-2t^2 - t + 1) + B(2t^2 - 3t + 1) + C(1-t^2)$ Comp. the coefficient of t^2, t and constant term $0 = -2A + 2B - C \quad C = -2A + 2B$ $0 = -A - 3B$ $1 = A + B + C$ $\therefore\ 1 = -A + 3B$ $0 = -A - 3B$ $1 = -2A$ $\therefore\ A = \frac{-1}{2}, B = \frac{1}{6}$ and $C = \frac{4}{3}$ $\therefore I = -\int \frac{-1}{2(1-t)} + \frac{1}{6(1+t)} + \frac{4}{3(1-2t)} dt$ $= \left[\frac{+1}{2}\log\mid 1-t\mid + \frac{1}{6}\log\mid 1+t\mid + \frac{4}{3}\log\mid 1-2t\mid . \left(\frac{-1}{2}\right)\right] + c$ $= \frac{-1}{2}\log\mid 1-t\mid - \frac{1}{6}\log\mid 1+t\mid + \frac{2}{3}\log\mid 1-2t\mid + c$ replacing t $= I = \frac{-1}{2}\mid 1-\cos x\mid - \frac{1}{6}\log\mid 1+\cos x\mid + \frac{2}{3}\log\mid 1-\cos x\mid + c$ ans.
→	**<u>Type : 2</u> Linear and Quadratic Faction**
Q.5)	(a)$I = \int \frac{x}{(x-1)(x^2+4)} dx$ (b)$I = \int \frac{1}{1+x+x^2+x^3} dx$
Sol.5)	(a)$I = \int \frac{x}{(x-1)(x^2+4)} dx$ let $\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4} dx$ $\Rightarrow\ x = A(x^2+4) + (Bx+c)(x-1)$ $\Rightarrow\ x = A(x^2+4) + (Bx^2 - Bx + Cx - C)$

Comp. the coefficient of x^2 , x and constant term

$0 = A + B$

$1 = -B + C$

$0 = 4A - C$

Solving these equations,

we get $A = \frac{1}{5}. B = \frac{-1}{5}$ and $C = \frac{4}{5}$

$\therefore I = \int \frac{1}{5(x-1)} + \frac{\frac{-1}{5}x + \frac{4}{5}}{x^2+4} dx$

$= \frac{1}{5}\int \frac{1}{x-1} dx - \frac{1}{5}\int \frac{x}{x^2+4} dx + \frac{4}{5}\int \frac{1}{x^2+4} dx$

put $x^2 + 4 = t$

$\therefore\ x\,dx = \frac{dt}{2}$

$\therefore I = \frac{1}{5}\log|x-1| - \frac{1}{10}\int \frac{dt}{t} + \frac{4}{5} \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

$I = \frac{1}{5}\log|x-1| - \frac{1}{10}\log|x^2+4| + \frac{2}{3}\tan^{-1}\left(\frac{x}{2}\right) + c$ ans.

(b) $I = \int \frac{1}{1+x+x^2+x^3} dx$

$= \int \frac{1}{(1+x)+x^2(1+x)} dx$

$= \int \frac{1}{(1+x)(1+x^2)} dx$

let $\frac{1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{Bx+c}{x^2+1}$

$1 = A(x^2+1) + (Bx + C)(x+1)$

$1 = A(x^2+1) + (Bx^2 + Bx + Cx + C)$

Comp. the coefficient of x^2 , x and constant

$0 = A + B \Rightarrow B = -A$

$0 = B + C \quad \therefore 0 = -A + C$

$1 = A + C \qquad 1 = A + C$

$1 = 2C$

$C = \frac{1}{2}, A = \frac{1}{2}$ and $B = \frac{-1}{2}$

$\therefore I = \frac{1}{2(x+1)} + \frac{\frac{-1}{2}+\frac{1}{2}}{x^2+1} dx$

$= \frac{1}{2}\int \frac{1}{x+1} dx - \frac{1}{2}\int \frac{x}{x^2+1} dx + \frac{1}{2}\int \frac{1}{x^2+1} dx$

put $x^2 + 1 = t$

$dx = \frac{dt}{2}$

$= \frac{1}{2}\log|x+1| - \frac{1}{4}\int \frac{dt}{t} + \frac{1}{2}\tan^{-1}x$

$I = \frac{-1}{2}\log|x+1| - \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + c$ ans.

Q.6) $I = \int \frac{x}{x^3-1} dx$

Sol.6) $I = \int \frac{x}{x^3-1} dx$

	$= \int \frac{x}{(1-x)(x^2+x+1)} dx$ let $\frac{x}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+c}{x^2+x+1}$ $x = A(x^2 + x + 1) + (Bx + c)(x - 1)$ $x = A(x^2 + x + 1) + (Bx - Bx^2 + cx + c)$ Comp. the coefficient of x^2 , x and constant term $0 = A + B \Rightarrow B = -A$ $1 = A - B + C \Rightarrow 1 = 2A + C$ $0 = A - C \Rightarrow 0 = A - C$ $1 = 3A$ $A = \frac{1}{3}, B = \frac{-1}{3}$ and $C = \frac{1}{3}$ $\therefore I = \int \frac{1}{3(x-1)} + \frac{\frac{-1}{3}x+\frac{1}{3}}{x^2+x+1} dx$ $= \frac{1}{3}\int \frac{1}{x-1} dx - \frac{1}{3}\int \frac{x-1}{x^2+x+1} dx \quad \left\{type\int \frac{linear}{quadratic}\right\}$ $= I = \frac{1}{3}\log \mid x-1 \mid - \frac{1}{3}I$ ....(1) where $I = \int \frac{x-1}{x^2+x+1} dx$ Proceed Yourself and get I $I = \frac{1}{3}\log \mid x-1 \mid - \frac{1}{6}\log \mid x^2+x+1 \mid + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x}{\sqrt{3}}\right)$
Q.7)	(a)$I = \int \frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta} d\theta$
Sol.7)	(a)$I = \int \frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta} d\theta$ $= \int \frac{\tan\theta(1+\tan^2\theta)}{1+\tan^3\theta} d\theta$ $= \int \frac{\tan\theta.sec^2\theta}{1+\tan^3\theta} d\theta$ put $\tan\theta = t$ $sec^2\theta\, d\theta = dt$ $\therefore I = \int \frac{t}{1+t^3} dt$ $= \int \frac{t}{(1+t)(t^2-t+1)} dt$ Proceed as above Qns. :- $\frac{-1}{3}\log \mid 1+\tan\theta \mid + \frac{1}{6}\log \mid \tan^2\theta - \tan\theta + 1 \mid + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)$ ans.
→	**Type : 3 Linear and Linear repeating factors**
Q.8)	(a)$I = \int \frac{3x+1}{(x+2)(x+2)(x-2)^2} dx$ (b)$I = \int \frac{x^2+x+1}{(x-1)^3} dx$
Sol.8)	(a)$I = \int \frac{3x+1}{(x+2)(x+2)(x-2)^2} dx$ let $\frac{3x+1}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$ $3x + 1 = A(x-2)^2 + B(x-2)(x+2) + c(x+2)$ $3x + 1 = A(x^2 - 4x + 4) + B(x^2 - 4) + C(x + 2)$

	Comp. the coefficient of x^2, x and constant $0 = A + B \quad \Rightarrow \quad B = -A$ $3 = -4A + C \quad \Rightarrow \quad 3 = -4A + c$ $1 = 4A - 4B + 2C \Rightarrow 1 = 8A + 2C$ solving these equations, we get $A = \frac{-5}{16}, B = \frac{5}{16}$ and $C = \frac{7}{4}$ $\therefore I = \frac{-5}{16(x+2)} + \frac{5}{16(x-2)} + \frac{7}{4(x-2)^2} dx$ $I = \frac{-5}{16} \log \mid x+2 \mid + \frac{5}{16} \log \mid x-2 \mid - \frac{7}{4(x-2)} + c$ ans. .....$\left\{ \text{Since} \int \frac{1}{x^2} dx = \frac{-1}{x} \right\}$ (b)$I = \int \frac{x^2+x+1}{(x-1)^3} dx$ let $\frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$ $x^2 + x + 1 = A(x-1)^2 + B(x-1) + C$ $x^2 + x + 1 = A(x^2 - 2x + 1) + B(x-1) + C$ Comp. the coefficient of x^2 , x and constant $1 = A$ $1 = -2A + B$ $1 = A - B + C$ solving these equation we get $A = 1, B = 3, C = 3$ $\therefore I = \int \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3} dx$ $= \log \mid x-1 \mid - \frac{3}{(x-1)} + 3\int (x-1)^{-3} dx$ $= \log \mid x-1 \mid - \frac{3}{x-1} + 3 \frac{(x-1)^{-2}}{-2} + c$ $\therefore I = \log \mid x-1 \mid - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + c$ ans.
Q.9)	$I = \int \frac{3x+5}{x^3-x^2-x+1} dx$
Sol.9)	$I = \int \frac{3x+5}{x^3-x^2-x+1} dx$ $= \int \frac{3x+5}{x^2(x-1)-1(x-1)} dx$ $= \int \frac{3x+5}{(x-1)(x^2-1)} dx$ $= \int \frac{3x+5}{(x-1)(x+1)(x-1)} dx$ $= \int \frac{3x+5}{(x+1)(x-1)^2} dx$ let $\frac{3x+5}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ $3x + 5 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$ $3x + 5 = A(x^2 - 2x + 1) + B(x^2 - 1) + C(x+1)$ $0 = A + B$ $3 = -2A + C$ $5 = A - B + C$

	solving these equation , we get $A = \frac{1}{2}, B = \frac{-1}{2}, C =$ $\therefore\ I = \int \frac{1}{2(x+1)} - \frac{1}{2(x-1)} + \frac{4}{(x-1)^2} dx$ $I = \frac{1}{2}\log \mid x+1 \mid - \frac{1}{2}\log \mid x-1 \mid - \frac{4}{x-1} + c$ ans.
→	**Type : 4 Even Power of x let x^2 = y (temp.)**
Q.10)	(a)$I = \int \frac{x^2}{(x^2+1)(x+4)} dx$ (b)$I = \int \frac{1}{(x^4-1)} dx$
Sol.10)	(a)$I = \int \frac{x^2}{(x^2+1)(x^{+4})} dx$ let $x^2 = y$ $\therefore \frac{x^2}{(x^2+1)(x^2+4)} = \frac{y}{(y+1)(y+4)}$ let $\frac{y}{(y+1)(y+4)} = \frac{A}{y+1} + \frac{B}{y+4}$ $y = A(y+4) + B(y+1)$ Comp. coefficient of y and constant $1 = A + B$ $0 = 4A + B$ $1 = -3A$ $A = \frac{-1}{3} \therefore . B = \frac{4}{3}$ $\therefore I = \int \frac{-1}{3(x^2+1)} + \frac{4}{3(x^2+4)} dx$ $= \frac{-1}{3} \int \frac{1}{(x^2+1)} dx + \frac{4}{3} \int \frac{1}{x^2+(2)^2} dx$ $= \frac{-1}{3} \tan^{-1} x + \frac{4}{3} \times \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$ ans. (b)$I = \int \frac{1}{(x^4-1)} dx$ $= \int \frac{1}{(x^2+1)(x^2-1)} dx$ let $x^2 = y$ $\therefore \frac{1}{(x^2+1)(x^2-1)} = \frac{1}{(y+1)(y-1)}$ let $\frac{1}{(y+1)(y-1)} = \frac{A}{y+1} + \frac{B}{y-1}$ $1 = A(y-1) + B(y+1)$ Comp. $0 = A + B$ $1 = -A + B$ $1 = 2B$ $\therefore B = \frac{1}{2}$ and $A = \frac{-1}{2}$ $\therefore I = \int \frac{-1}{2(x^2+1)} + \frac{1}{2(x^2-1)} dx$ $= \frac{-1}{2} \int \frac{1}{x^2+1} dx + \frac{1}{2} \int \frac{1}{x^2-1} dx$

	$= \frac{-1}{2}\tan^{-1}x + \frac{1}{2} \times \frac{1}{2\times 1}\log\left\|\frac{x-1}{x+1}\right\| + c$ $= \frac{-1}{2}\tan^{-1}x + \frac{1}{4}\log\left\|\frac{x-1}{x+1}\right\| + c$ ans.