

Determinants

Class 12th

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| | Solve for x |
| Q.1) | Solve $\begin{vmatrix} a + xa - xa - x \\ a - xa + xa - x \\ a - xa + xa + x \end{vmatrix} = 0$ |
| Sol.1) | $c_1 \rightarrow c_1 + c_2 + c_3$ $\Rightarrow \begin{vmatrix} 3a - xa - xa - x \\ 3a - xa + xa - x \\ 3a - xa - xa + x \end{vmatrix} = 0$ <p>taking $(3a - x)$ common from</p> $\Rightarrow (3a - x) \begin{vmatrix} 1a - xa - x \\ 1a + xa - x \\ 1a - xa + x \end{vmatrix} = 0$ <p>$R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$</p> $\Rightarrow (3a - x) \begin{vmatrix} 1a - xa - x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$ <p>taking $2x$ common from R_2 & R_3 both</p> $\Rightarrow (3a - x)(2x)(2x) \begin{vmatrix} 1a - xa - x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$ <p>expanding along R_1</p> $\Rightarrow (3a - x)(4x^2)[1(1) - 0 + 0] = 0$ $\Rightarrow (3a - x)(4x^2) = 0$ $\Rightarrow x = 3a, x = 0 \quad \text{ans.}$ |
| Q.2) | Solve $\begin{vmatrix} x - 22x - 33x - 4 \\ x - 42x - 93x - 16 \\ x - 82x - 273x - 64 \end{vmatrix} = 0$ |
| Sol.2) | <p>We have $\begin{vmatrix} x - 22x - 33x - 4 \\ x - 42x - 93x - 16 \\ x - 82x - 273x - 64 \end{vmatrix} = 0$</p> <p>$R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$</p> $\Rightarrow \begin{vmatrix} x - 22x - 33x - 4 \\ -2 - 6 - 12 \\ -6 - 24 - 60 \end{vmatrix} = 0$ <p>$c_2 \rightarrow c_2 - 2c_1$ and $c_3 \rightarrow c_3 - 3c_1$</p> $\Rightarrow \begin{vmatrix} x - 212 \\ -2 - 2 - 6 \\ -6 - 12 - 42 \end{vmatrix} = 0$ <p>taking (-2) and (-6) common from R_2 & R_3 respectively</p> $\Rightarrow 12 \begin{vmatrix} x - 212 \\ 113 \\ 127 \end{vmatrix} = 0$ <p>expanding along R_1</p> $\Rightarrow 12[(x - 2)(7 - 6) - 1(7 - 3) + 2(2 - 1)] = 0$ $\Rightarrow 12[x - 2 - 4 + 2] = 0$ |

$$\Rightarrow 12(x - 4) = 0$$

$$\Rightarrow x = 4 \quad \text{ans.}$$

Proving Questions

Q.3) Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a)$

Sol.3) let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$

$$R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b - a & b^2 - a^2 \\ 0 & c - a & c^2 - a^2 \end{vmatrix}$$

taking (b-a) and (c-a) common from R_2 & R_3 respectively

$$= (b - a)(c - a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b + a \\ 0 & 1 & c + a \end{vmatrix}$$

expanding along R_1

$$= (b - a)(c - a)[1(c + a - b - a) - a(0) + a^2(0)]$$

$$= (b - a)(c - a)(c - b)$$

$$= (a - b)(b - c)(c - a) = \text{RHS} \quad \text{Proved}$$

Q.4) Show that $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(ab + bc + ca)$

Sol.4) let $\Delta = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

$$c_1 \rightarrow ac_1; c_2 \rightarrow bc_2 \text{ and } c_3 \rightarrow ac_3$$

$$= \frac{1}{abc} \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$$

taking abc common from C_3

$$= \frac{abc}{abc} \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$c_2 \leftrightarrow c_3$$

$$= - \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$c_1 \leftrightarrow c_2$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$c_2 \rightarrow c_2 - c_1 \text{ and } c_3 \rightarrow c_3 - c_1$$

$$= \begin{vmatrix} 1 & & 0 \\ a^2 & b^2 - a^2 & c^2 - a^2 \\ a^3 & (b - a)(b^2 + ab + a^2) & (c - b)(c^2 + ac + a^2) \end{vmatrix}$$

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| | <p>taking $(b-a)$ and $(c-b)$ common from c_2 & c_3</p> $= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b+a & c+a \\ a^3 & b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix}$ <p>expanding along R_1</p> $= (b-a)(c-a)[(b+a)(c^2+ac+a^2) - (c+a)(b^2+ab+a^2)]$ $= (b-a)(c-a)[bc^2+abc+a^2b+ac^2+a^2c+a^3 - b^2c-abc-a^2c-ab^2-a^2b-a^3]$ $= (b-a)(c-a)(bc^2+ac^2-b^2c-ab^2)$ $= (b-a)(c-a)[bc(c-b)+a(c^2-b^2)]$ $= (b-a)(c-a)(c-b)[bc+a(c+b)]$ $= (b-a)(c-a)(c-b)(bc+ac+ab)$ $= (a-b)(b-c)(c-a)(ab+bc+ac) = \text{RHS}$ |
| Q.5) | <p>Show that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$</p> |
| Sol.5) | <p>$c_1 \rightarrow c_1 + c_2 + c_3$</p> $= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$ <p>taking $2(a+b+c)$ common from C_1</p> $= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$ <p>$R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$</p> $= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+2a & b \\ 0 & a & c+a+2b \end{vmatrix}$ <p>expanding along R_1</p> $= 2(a+b+c)[(a+b+c)(a+b+c)] = 2(a+b+c)^3 \quad \text{ans.}$ |
| Q.6) | <p>Show $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$</p> |
| Sol.6) | <p>$R_3 \rightarrow R_3 + R_1$</p> $= \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix}$ <p>taking $(\alpha+\beta+\gamma)$ common from R_3</p> $= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix}$ <p>$c_2 \rightarrow c_2 - c_1$ and $c_3 \rightarrow c_3 - c_1$</p> $= (\alpha+\beta+\gamma) \begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix}$ <p>taking $(\beta-\alpha)$ and $(\gamma-\alpha)$ common from C_2 & C_3 respectively</p> |

$$= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha) \begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta - \alpha & \gamma - \alpha \\ 1 & 0 & 0 \end{vmatrix}$$

expanding

$$\begin{aligned} &= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)[\alpha(0) + 1(\gamma + \alpha) + 1(-\beta - \alpha)] \\ &= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma + \alpha - \beta - \alpha) \\ &= (\alpha + \beta + \gamma)(\beta - \alpha)(\gamma - \alpha)(\gamma - \beta) \\ &= (\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)(\alpha + \beta + \gamma) = \text{RHS} \end{aligned}$$

Q.7)

Show $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$

Sol.7)

$$\begin{aligned} R_1 &\rightarrow R_1 + R_2 + R_3 \\ &= \begin{vmatrix} 2(a+b+c) & 0 & a+b+c \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} \end{aligned}$$

taking $(a+b+c)$ common from R_1

$$= (a+b+c) \begin{vmatrix} 2 & 0 & 1 \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$$

$$c_1 \rightarrow c_1 - 2c_3$$

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ c+a-2b & b-c & b \\ a+b-2c & c-a & c \end{vmatrix}$$

expanding along R_1

$$\begin{aligned} &= (a+b+c)[1(c+a-2b)(c-a) - (a+b-2c)(b-c)] \\ &= (a+b+c)[c^2 - ac + ac - a^2 - 2bc + 2ab - ab + ac - b^2 + bc + 2bc - 2c^2] \\ &= (a+b+c)(-a^2 - b^2 - c^2 + ab + bc + ca) \\ &= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= -[a^3 + b^3 + c^3 - 3abc] \\ &= 3abc - a^3 - b^3 - c^3 = \text{RHS} \quad \text{ans.} \end{aligned}$$

Q.8)

Show $\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (x^3 - 1)^2$

Sol.8)

$$\begin{aligned} c_1 &\rightarrow c_1 + c_2 + c_3 \\ &= \begin{vmatrix} 1+x+x^2 & x & x^2 \\ 1+x+x^2 & 1 & x \\ 1+x+x^2 & x^2 & 1 \end{vmatrix} \end{aligned}$$

taking $(1+x+x^2)$ common from C_1

$$= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

$$R \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$$

$$= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 1 & x^2-x & 1-x^2 \end{vmatrix}$$

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| | $= (1 + x + x^2) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 - x & x(1 - x) \\ 0 & -x(1 - x) & (1 + x)(1 - x) \end{vmatrix}$ <p>taking $(1 - x)$ common from R_2 & R_3 both</p> $= (1 + x + x^2)(1 - x)^2 \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & x \\ 0 & -x & 1 + x \end{vmatrix}$ <p>expanding</p> $= (1 + x + x^2)(1 - x)^2 [1 + x + x^2]$ $= (1 - x)^2 (1 + x + x^2)^2$ $= [(1 - x)(1 + x + x^2)]^2$ $= (1 - x^3)^2 \quad \dots\{a^3 - b^3 = (a - b)(a^2 + ab + b^2)\}$ $= (x^3 - 1)^2 = \text{RHS} \quad \dots\{(a - b)^2 = (b - a)^2\}$ |
| Q.9) | <p>Show $\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & b^2 \end{vmatrix} = (a^3 + b^3)^2$</p> |
| Sol.9) | $c_1 \rightarrow c_1 + c_2 + c_3$ $= \begin{vmatrix} a^2 + 2ab + b^2 & 2ab & b^2 \\ a^2 + 2ab + b^2 & a^2 & 2ab \\ a^2 + 2ab + b^2 & b^2 & a^2 \end{vmatrix}$ $= (a + b)^2 \begin{vmatrix} 1 & 2ab & b^2 \\ 1 & a^2 & 2ab \\ 1 & b^2 & a^2 \end{vmatrix}$ $R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1$ $= (a + b)^2 \begin{vmatrix} 1 & 2ab & b^2 \\ 0 & a^2 - 2ab & 2ab - b^2 \\ 0 & b^2 - 2ab & a^2 - b^2 \end{vmatrix}$ <p>expanding along R_1</p> $= (a + b)^2 [(a^2 - 2ab)(a^2 - b^2) - (b^2 - 2ab)(2ab - b^2)]$ $= (a + b)^2 [a^4 - a^2b^2 - 2a^3b + 2ab^3 - 2ab^3 + b^4 + 2a^2b^2 - 2ab^3]$ $= (a + b)^2 [a^4 + b^4 + a^2b^2 - 2a^3b + 2a^2b^2 - 2ab^3]$ $= (a + b)^2 (a^2 - ab + b^2)^2 \quad \dots\{(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca\}$ $= [(a + b)(a^2 - ab + b^2)]^2$ $= (a^3 + b^3)^2 \quad \text{ans.}$ |
| Q.10) | <p>Show $\begin{vmatrix} a^2 + 1abac \\ abb^2 + 1bc \\ cacbc^2 + 1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$</p> |
| Sol.10) | $R_1 \rightarrow aR_1, R_2 \rightarrow bR_2 \text{ and } R_3 \rightarrow cR_3$ $= \frac{1}{abc} \begin{vmatrix} a^3 + aa^2ba^2c \\ ab^2b^3 + bb^2c \\ c^2ac^2bc^3 + c \end{vmatrix}$ <p>taking a, b, c common from c_1, c_2 & c_3 respectively</p> $= \frac{abc}{abc} \begin{vmatrix} a^2 + 1a^2a^2 \\ b^2b^2 + 1b^2 \\ c^2c^2c^2 + 1 \end{vmatrix}$ $R_1 \rightarrow R_1 + R_2 + R_3$ |

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2b^2+1b^2 & & \\ c^2c^2c^2+1 & & \end{vmatrix}$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 1 & 1 \\ b^2b^2+1b^2 & & \\ c^2c^2c^2+1 & & \end{vmatrix}$$

$$c_2 \rightarrow c_2 - c_1 \text{ and } c_3 \rightarrow c_3 - c_1$$

$$= (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 0 & 0 \\ b^2b^2+1b^2 & & \\ c^2c^2c^2+1 & & \end{vmatrix}$$

expanding

$$= (1+a^2+b^2+c^2)[1] = 1+a^2+b^2+c^2 = \text{RHS}$$