

DEFINITE INTEGRALS

Q.1)	$I = \int_3^4 e^{3x} + 2x \, dx$
Sol.1)	$a = 3, b = 4, nh = 4 - 3 = 1$ $I = \lim_{h \rightarrow 0} h[f(3) + f(3+h) + f(3+2h) + \dots + f(3+(n-1)h)]$ $= \lim_{h \rightarrow 0} h[(e^9 + 6) + (e^{9+3h} + 6 + 2h) + (e^{9+6h} + 6 + 4h) + \dots + (e^{9+3(n-1)h} + 6 + 2(n-1)h)]$ $= \lim_{h \rightarrow 0} h[\{e^9 + e^{9+3h} + e^{9+6h} + \dots + e^{9+3(n-1)h}\} + \{(6 + 6 + 6 + \dots \dots n \text{ term}) + (2h + 4h + \dots \dots 2(n-1)h)\}]$ $= \lim_{h \rightarrow 0} h[e^9 + e^{9+3h} + e^{9+6h} + \dots \dots e^{9+3(n-1)h}] + \lim_{h \rightarrow 0} h[6n + 2h(1 + 2 + \dots \dots (n-1))]$ $= e^9 \lim_{h \rightarrow 0} h[1 + e^{3h} + e^{6h} + \dots \dots e^{3(n-1)h}] + \lim_{h \rightarrow 0} h\left[6n + \frac{2h \cdot n(n-1)}{2}\right]$ $= e^9 \lim_{h \rightarrow 0} h\left[1 \left(\frac{e^{3h} - 1}{e^{3h} - 1}\right)\right] + \lim_{h \rightarrow 0} [6nh + (nh)(nh - h)]$ $= e^9 \lim_{h \rightarrow 0} h\left[\frac{e^{3nh} - 1}{\frac{e^{3nh} - 1}{3h} \times 3h}\right] + \lim_{h \rightarrow 0} [6n + (nh)(nh - h)]$ <p>Put $nh = 1$</p> $\therefore I = e^9 \frac{(e^3 - 1)}{\lim_{h \rightarrow 0} \left(\frac{e^{3h} - 1}{3h}\right) \times 3} + \lim_{h \rightarrow 0} [6 + (1)(1 - h)]$ $I = \frac{e^9(e^3 - 1)}{3} + 6 + 1$ $I = \frac{e^9(e^3 - 1)}{3} + 7 \quad \text{Ans.....}$
Q.2)	Evaluate $\int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x}\right) dx$
Sol.2)	$I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - \sin x}{1 - \cos x}\right) dx$ $I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1 - 2 \sin \frac{x}{2} \cdot \cos \frac{x}{2}}{2 \sin^2 \frac{x}{2}}\right) dx$ $I = \int_{\frac{\pi}{2}}^{\pi} e^x \left(\frac{1}{2} \operatorname{cosec}^2 \left(\frac{x}{2}\right) - \cot \left(\frac{x}{2}\right)\right) dx$ $I = \int_{\frac{\pi}{2}}^{\pi} e^x \cot \left(\frac{x}{2}\right) dx + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} e^x \cdot \operatorname{cosec}^2 \left(\frac{x}{2}\right) dx$ $I = - \left[\left(\cot \frac{x}{2} \cdot e^x\right)_{\pi/2}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} -\frac{1}{2} \cdot \operatorname{cosec}^2 \left(\frac{x}{2}\right) e^x dx \right] + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} e^x \operatorname{cosec}^2 \frac{x}{2}$ $I = - \left[\cot \frac{\pi}{2} \cdot e^x - \cot \frac{\pi}{4} \cdot e^{\frac{\pi}{2}} \right] - \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} e^x \operatorname{cosec}^2 \frac{x}{2} dx + \frac{1}{2} \int_{\frac{\pi}{2}}^{\pi} e^x \operatorname{cosec}^2 \left(\frac{x}{2}\right) dx$ $I = - \left[0 - e^{\frac{\pi}{2}} \right] \quad \dots \dots \dots \left\{ \because \cot \left(\frac{\pi}{2}\right) = 0, \cot \left(\frac{\pi}{4}\right) = 1 \right\}$

	$\therefore I = e^{\frac{\pi}{2}}$ <i>Ans</i>
Q.3)	$I = \int_0^{\frac{\pi}{2}} \sin(2x) \cdot \tan^{-1}(\sin x) dx$
Sol.3)	$I = 2 \int_0^{\frac{\pi}{2}} \sin x \cdot \cos x \cdot \tan^{-1}(\sin x) dx$ <p>Put $\sin x = t$ when $x = 0$; $t = 0$ $\cos x dx = dt$ when $x = \frac{\pi}{2}$; $t = 1$</p> $\therefore I = 2 \int_0^1 t \cdot \tan^{-1}(t) dt$ $= 2 \left[\left(\tan^{-1} t \cdot \frac{t^2}{2} \right)_0^1 - \int_0^1 \frac{1}{1+t^2} \cdot \frac{t^2}{2} dt \right]$ $= 2 \left[\left(\frac{\pi}{4} \cdot \frac{1}{2} \right) - 0 - \frac{1}{2} \int_0^1 \frac{t^2}{1+t^2} dt \right]$ $= 2 \left[\frac{\pi}{8} - \frac{1}{2} \int_0^1 \frac{1+t^2-1}{1+t^2} dt \right]$ $= \frac{\pi}{4} - \int_0^1 1 - \frac{1}{1+t^2} dt$ $= \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$ $= \frac{\pi}{4} - \left[\left(1 - \frac{\pi}{4} \right) - 0 \right]$ $I = \frac{\pi}{2} - 1 \quad \text{Ans....}$
Q.4)	$I = \int_0^{\frac{\pi}{2}} \frac{x + \sin x}{1 + \cos x} dx$
Sol.4)	$I = \int_0^{\frac{\pi}{2}} \frac{x + 2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)}{2 \cos^2\left(\frac{x}{2}\right)} dx$ $I = \int_0^{\frac{\pi}{2}} \frac{x}{2 \cos^2(x/2)} + \frac{2 \sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)}{2 \cos^2\left(\frac{x}{2}\right)} dx$ $I = \int_0^{\frac{\pi}{2}} \frac{1}{2} x \sec^2\left(\frac{x}{2}\right) + \tan\left(\frac{x}{2}\right) dx$ $I = \frac{1}{2} \int_0^{\frac{\pi}{2}} x \cdot \sec^2\left(\frac{x}{2}\right) dx + \int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx$ $I = \frac{1}{2} \left[\left(x \cdot \tan\left(\frac{x}{2}\right) \cdot 2 \right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 1 \cdot \tan\left(\frac{x}{2}\right) \cdot 2 dx \right] + \int_0^{\frac{\pi}{2}} \tan\left(\frac{x}{2}\right) dx$ $I = \frac{1}{2} \left[\left(\frac{\pi}{2} \cdot \tan \frac{\pi}{4} \cdot 2 \right) - 0 \right] - \int_0^{\frac{\pi}{2}} \tan \frac{x}{2} dx + \int_0^{\frac{\pi}{2}} \tan \frac{x}{2} dx$ $I = \frac{1}{2} \left[\frac{\pi}{1} \right] = \frac{\pi}{2} \quad \text{Ans}$
Q.5)	$I = \int_0^{\frac{\pi}{4}} \sin^3(2t) \cdot \cos(2t) dt$
Sol.5)	<p>(a) $I = \int_0^{\frac{\pi}{4}} \sin^3(2t) \cdot \cos(2t) dt$</p> <p>Put $\sin(2t) = z$</p> $\therefore 2 \cos(2t) dt = dz \quad \text{when } t = 0 ; z = 0$

	$\cos(2t) dt = \frac{dz}{2} \quad \text{when } t = \frac{\pi}{4}; z = 1$ $\therefore I = \frac{1}{2} \int_0^1 z^3 dz$ $= \frac{1}{2} \left(\frac{z^4}{4} \right)_0^1$ $= \frac{1}{2} \left(\frac{1}{4} - 0 \right)$ $I = \frac{1}{8} \quad \text{Ans.....}$
Q.6)	$I = \int_4^9 \frac{\sqrt{x}}{(30 - e^{\frac{3}{2}x})^2} dx$
Sol.6)	$I = \int_4^9 \frac{\sqrt{x}}{(30 - e^{\frac{3}{2}x})^2} dx$ <p>Put $30 - x^{3/2} = t$ when $x = 4$; $t = 30 - 8 = 22$</p> $\frac{-3}{2} x^{1/2} dx = dt \quad \text{when } x = 9 ; t = 30 - 27 = 3$ $\sqrt{x} dx = \frac{-2}{3} dt$ $\therefore I = \frac{-2}{3} \int_{22}^3 \frac{dt}{t^2}$ $= \frac{-2}{3} \left(-\frac{1}{t} \right)_{22}^3$ $= \frac{2}{3} \left[\frac{1}{3} - \frac{1}{22} \right]$ $= \frac{2}{3} \left(\frac{22-3}{66} \right) = \frac{2(19)}{3 \times 66} = \frac{19}{99} \quad \text{Ans....}$
Q.7)	$\int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cos^5 \phi d\phi$
Sol.7)	$I = \int_0^{\frac{\pi}{2}} \sqrt{\sin \phi} \cdot \cos^4 \phi \cdot \cos \phi d\phi$ <p>Put $\sin \phi = t$ when $\phi = 0$; $t = 0$</p> $\cos \phi d\phi = dt \quad \text{when } \phi = \frac{\pi}{2} ; t = 1$ $\therefore I = \int_0^1 \sqrt{t} (1 - \sin^2 \phi)^2 \cdot dt$ $= \int_0^1 \sqrt{t} (1 - t^2)^2 dt$ $= \int_0^1 \sqrt{t} (1 + t^4 - 2t^2) dt$ $= \int_0^1 \sqrt{t} + t^{9/2} - 2t^{5/2} dt$ $= \left(\frac{2}{3} t^{3/2} + \frac{2}{11} t^{11/2} - 2 \cdot \frac{2}{7} t^{7/2} \right)_0^1$ $= \left(\frac{2}{3} + \frac{2}{11} - \frac{4}{7} \right) - (0)$ $= \frac{154+42-132}{231}$

	$= \frac{64}{231} \quad \text{Ans ...}$
Q.8)	$I = \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$
Sol.8)	$I = \int_0^1 \sin^{-1} \left(\frac{2x}{1+x^2} \right) dx$ $I = 2 \int_0^1 \tan^{-1} x \, dx \quad \dots \dots \left\{ \because 2 \tan^{-1} x = \sin^{-1} \left(\frac{2x}{1+x^2} \right) \right\}$ $I = 2 \int_0^1 \tan^{-1} x \cdot 1 \, dx$ $= 2 \left[(\tan^{-1} x \cdot x)_0^1 - \int_0^1 \frac{1}{1+x^2} \cdot x \, dx \right]$ <p>Put $1 + x^2 = t$ when $x = 0$; $t = 1$</p> $x \, dx = \frac{dt}{2} \quad x = 1, t = 2$ $\therefore I = 2 \left[\left(\frac{\pi}{4} \cdot 1 \right) - (0) - \frac{1}{2} \int_1^2 \frac{dt}{t} \right]$ $= 2 \left[\frac{\pi}{4} - \frac{1}{2} (\log t)_1^2 \right]$ $= 2 \left[\frac{\pi}{4} - \frac{1}{2} (\log 2 - \log 1) \right]$ $= 2 \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] \quad \dots \dots \left\{ \because \log 1 = 0 \right\}$ $I = \frac{\pi}{2} - \log 2 \quad \text{Ans ...}$
Q.9)	$I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} dx$
Sol.9)	$I = \int_0^1 \frac{1}{\sqrt{1+x}-\sqrt{x}} dx$ <p><i>Rationalize</i></p> $I = \int_0^1 \frac{\sqrt{1+x}+\sqrt{x}}{1+x-x} dx$ $I = \left[\frac{2}{3} (1+x)^{\frac{3}{2}} + \frac{2}{3} x^{\frac{3}{2}} \right]_0^1$ $= \frac{2}{3} \left[(2)^{\frac{3}{2}} + (1)^{\frac{3}{2}} \right] - \frac{2}{3} \left[(1)^{\frac{3}{2}} + 0 \right]$ $= \frac{2}{3} (2\sqrt{2} + 1) - \frac{2}{3} (1)$ $= \frac{4\sqrt{2}}{3} + \frac{2}{3} - \frac{2}{3}$ $I = \frac{4\sqrt{2}}{3} \quad \text{ans.}$
Q.10)	$I = \int_0^1 \frac{2x+3}{5x^2+1} dx$
Sol.10)	$I = \int_0^1 \frac{2x+3}{5x^2+1} dx \quad (\text{separate})$ $I = 2 \int_0^1 \frac{x}{5x^2+1} dx + 3 \int_0^1 \frac{1}{5x^2+1} dx$ <p>Put $5x^2 + 1 = t$ when $x = 0$; $t = 1$</p> $10x \, dx = dt \quad \text{when } x = 1 ; t = 6$

$$x dx = \frac{dt}{10}$$

$$\therefore I = \frac{2}{10} \int_1^6 \frac{dt}{t} + \frac{3}{5} \int_0^1 \frac{1}{x^2 + \left(\frac{1}{\sqrt{5}}\right)^2} dt$$

$$= \frac{1}{5} [\log t]_1^6 + \frac{3}{5} \times \sqrt{5} [\tan^{-1}(x\sqrt{5})]_0^1$$

$$= \frac{1}{5} \left[(\log 6 - \log 1) + \frac{3}{\sqrt{5}} (\tan^{-1}\sqrt{5} - \tan^{-1}0) \right]$$

$$I = \frac{1}{5} \log 6 + \frac{3}{\sqrt{5}} \tan^{-1}(\sqrt{5}) \quad \text{ans.}$$