SOLUTIONS

1. Given equation can be rewritten as

$$\frac{\frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x})} = \frac{dx - dy}{dx + dy}$$

Applying componendo and dividendo, we get

$$\frac{dy}{dx} = \frac{e^{-x}}{e^x} \Rightarrow dy = e^{-2x}dx \Rightarrow 2y = -e^{-2x} + C$$

(Integrating both sides)

$\Rightarrow 2ye^{2x} = C \cdot e^{2x} - 1$

2. $P(E) = P(X = 2) + P(X = 3) + P(X = 5) + P(X = 7)$
$= 0.23 + 0.12 + 0.20 + 0.07 = 0.62$

OR

$\because \sum_{x=0}^{4} P(X = x) = 1$

$\Rightarrow P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1$

$\Rightarrow k + 2k + 4k + 2k + k = 1$

$\Rightarrow 10k = 1 \Rightarrow k = 0.1$

$\therefore P(X \le 1) = P(X = 0) + P(X = 1)$

$\Rightarrow P(X \le 1) = k + 2k = 3k = 0.3$

3. Let $I = \int \frac{2x}{\sqrt[3]{x^2 + 1}} dx$

Put $x^2 + 1 = z \Rightarrow 2x\,dx = dz$

$$\therefore I = \int \frac{dz}{(z)^{1/3}} = \int z^{-1/3} dz = \frac{z^{(-1/3)+1}}{(-1/3)+1} + C$$

$$= \frac{3}{2}(x^2 + 1)^{2/3} + C$$

4. We have, $P(A) = 1/5$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A)\,P(B)$ ($\because$ A and B are independent)

$= 1/5 + P(B) - (1/5)\,P(B) = 2P(B) - 1/5$ (Given)

$\Rightarrow P(B) = 1/3$

5. $\frac{xdy}{dx} - y = x^4 - 3x$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{x^4 - 3x}{x} \Rightarrow \frac{dy}{dx} - \frac{1}{x} \cdot y = x^3 - 3$$

$$\therefore \text{I.F.} = e^{-\int \frac{1}{x} dx} = e^{-\log x} = e^{\log(x)^{-1}} = x^{-1} = \frac{1}{x}$$

6. We have, $y = 4 + 3x - x^2$, a parabola with vertex at $\left(\frac{3}{2}, \frac{25}{4}\right)$.

Putting $y = 0$, we get $x^2 - 3x - 4 = 0$

$\Rightarrow (x - 4)(x + 1) = 0 \Rightarrow x = -1$ or $x = 4$

$\therefore$ Required area $= \int_{-1}^{4} (4 + 3x - x^2)dx$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4} = \frac{125}{6} \text{ sq. units}$$

7. Let $I = \int_0^{\pi/2} 2\sin x \cdot \cos x \tan^{-1}(\sin x)dx$

Put $\sin x = t \Rightarrow \cos x\,dx = dt$

Also, $x = 0 \Rightarrow t = 0$ and $x = \frac{\pi}{2} \Rightarrow t = 1$

$$\therefore I = 2\int_0^1 t \times \tan^{-1} t \cdot dt$$

$$= 2\left[\frac{t^2}{2} \times \tan^{-1} t\right]_0^1 - 2\int_0^1 \frac{1}{1+t^2} \times \frac{t^2}{2} \cdot dt$$

$$= 2 \times \frac{1}{2} \times \tan^{-1}\left(\tan\frac{\pi}{4}\right) - \int_0^1 \frac{1+t^2-1}{1+t^2} \cdot dt$$

$$= 1 \times \frac{\pi}{4} - \int_0^1 \left(1 - \frac{1}{1+t^2}\right) \cdot dt = \frac{\pi}{4} - [t - \tan^{-1} t]_0^1$$

$$= \frac{\pi}{4} - 1 + \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{2\pi}{4} - 1 = \left(\frac{\pi}{2} - 1\right)$$

8. We have $|\vec{a}| = 2, |\vec{b}| = 1$ and $\vec{a} \cdot \vec{b} = 1$

Now, $(3\vec{a} - 5\vec{b}) \cdot (2\vec{a} + 7\vec{b})$

$= 6|\vec{a}|^2 + 21\,\vec{a} \cdot \vec{b} - 10\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$

$= 6|\vec{a}|^2 + 11\vec{a} \cdot \vec{b} - 35|\vec{b}|^2$

$= 6(2)^2 + 11(1) - 35(1)^2 = 24 + 11 - 35 = 0$

OR

Let $\vec{a} = 2\hat{i} - 3\hat{k}$ and $\vec{b} = 4\hat{j} + 2\hat{k}$

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a} \times \vec{b}|$.

$$\text{Now, } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -3 \\ 0 & 4 & 2 \end{vmatrix} = 12\hat{i} - 4\hat{j} + 8\hat{k}$$

$$\therefore |\vec{a} \times \vec{b}| = \sqrt{(12)^2 + (-4)^2 + (8)^2} = \sqrt{144 + 16 + 64}$$

$$= \sqrt{224} = 4\sqrt{14} \text{ sq. units.}$$

9. The given planes are

$2x + 3y + 4z = 5$ ...(i)

and $x + y + z = 1$ ...(ii)

Equation of any plane through the line of intersection of these planes is given by

$2x + 3y + 4z - 5 + \lambda(x + y + z - 1) = 0$

$\Rightarrow (2 + \lambda)x + (3 + \lambda)y + (4 + \lambda)z - (5 + \lambda) = 0$ ...(iii)

$\because$ This plane is $\perp$ to the plane $x - y + z = 0$

$\therefore (2 + \lambda)\cdot 1 + (3 + \lambda)\cdot(-1) + (4 + \lambda)\cdot 1 = 0$

$\Rightarrow 2 + \lambda - 3 - \lambda + 4 + \lambda = 0 \Rightarrow \lambda = -3$

$\therefore$ Equation of the required plane is

$(2 - 3)x + (3 - 3)y + (4 - 3)z - (5 - 3) = 0$

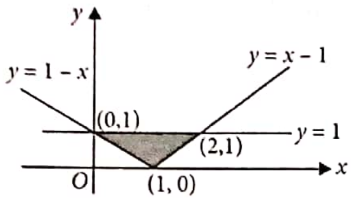
$\Rightarrow -x + z - 2 = 0 \Rightarrow x - z + 2 = 0$

Now distance of the point (1, 3, 6) from this plane

$$= \left|\frac{1-6+2}{\sqrt{1^2+(-1)^2}}\right| = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2} \text{ units}$$

10. We have, $y = x - 1$, if $x - 1 \geq 0$

$y = -x + 1$, if $x - 1 < 0$



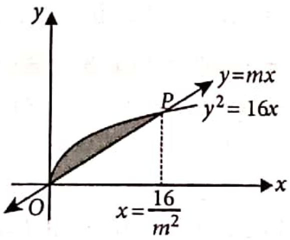
$\therefore$ Required area

$$= \int_0^2 1dx - \left[\int_0^1 (1-x)dx + \int_1^2 (x-1)dx\right]$$

$$= [x]_0^2 - \left[x - \frac{x^2}{2}\right]_0^1 - \left[\frac{x^2}{2} - x\right]_1^2 = 2 - \frac{1}{2} - \frac{1}{2} = 1 \text{ sq. unit}$$

OR

We have, $y^2 = 16x$, a parabola with vertex (0, 0) and line $y = mx$.



$\therefore$ Required area

$$= \int_0^{16/m^2} (\sqrt{16x} - mx)dx = \frac{2}{3}$$

$$\Rightarrow \left[4\times\frac{2}{3}x^{3/2} - m\left(\frac{x^2}{2}\right)\right]_0^{16/m^2} = \frac{2}{3}$$

$$\Rightarrow \frac{8}{3}\times\frac{64}{m^3} - \frac{m}{2}\frac{256}{m^4} = \frac{2}{3} \Rightarrow \frac{1}{m^3}\left[\frac{512}{3} - 128\right] = \frac{2}{3}$$

$\Rightarrow m = 4$

11. The sample space S of the given random experiment is

$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

Let A be the event that the die shows a number greater than 4 and B be the event that there is at least one tail.

$\therefore A = \{(T, 5), (T, 6)\}$

and $B = \{(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6), (H, T)\}$

$A \cap B = \{(T, 5), (T, 6)\}$

$\therefore P(B) = P\{(T, 1)\} + P\{(T, 2)\} + P\{(T, 3)\} + P\{(T, 4)\} + P\{(T, 5)\} + P\{(T, 6)\} + P\{(H, T)\}$

$$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{4} = \frac{3}{4}$$

$$P(A \cap B) = P\{(T,5)\} + P\{(T,6)\} = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\therefore \text{ Required probability } = P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{1/6}{3/4} = \frac{2}{9}$$

OR

Let A, B, C and E are respectively the events that a person is smoker and non-vegetarian, smoker and vegetarian, non-smoker and vegetarian, and the selected person is suffering from the disease.

Here, $n(A) = 160$, $n(B) = 100$,

$n(C) = 400 - (160 + 100) = 140$.

Also, $P(A) = \frac{160}{400}, P(B) = \frac{100}{400}, P(C) = \frac{140}{400}$

and $P(E/A) = \frac{35}{100}, P(E/B) = \frac{20}{100}, P(E/C) = \frac{10}{100}$

$\therefore$ Required probability

$$= P(A/E) = \frac{P(A)\cdot P(E/A)}{P(A)\cdot P(E/A) + P(B)\cdot P(E/B) + P(C)\cdot P(E/C)}$$

$$= \frac{\frac{160}{400}\times\frac{35}{100}}{\frac{160}{400}\times\frac{35}{100} + \frac{100}{400}\times\frac{20}{100} + \frac{140}{400}\times\frac{10}{100}}$$

$$= \frac{5600}{5600+2000+1400} = \frac{5600}{9000} = \frac{28}{45}$$

12. Here, $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}, \vec{b} = 2\hat{i} + \hat{j}, \vec{c} = 3\hat{i} - 4\hat{j} - 5\hat{k}$

$\therefore \vec{a} - \vec{b} = (\hat{i} + 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j}) = -\hat{i} + \hat{j} + \hat{k}$

$\vec{c} - \vec{b} = (3\hat{i} - 4\hat{j} - 5\hat{k}) - (2\hat{i} + \hat{j}) = \hat{i} - 5\hat{j} - 5\hat{k}$

Vector perpendicular to both $\vec{a} - \vec{b}$ and $\vec{c} - \vec{b}$ is

$$(\vec{a} - \vec{b}) \times (\vec{c} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 1 & -5 & -5 \end{vmatrix}$$

$$= (-5+5)\hat{i} - (5-1)\hat{j} + (5-1)\hat{k} = -4\hat{j} + 4\hat{k}$$

$\therefore$ Unit vector perpendicular to both $\vec{a} - \vec{b}$ and $\vec{c} - \vec{b}$

$$= \frac{-4\hat{j}+4\hat{k}}{|-4\hat{j}+4\hat{k}|} = \frac{-4\hat{j}+4\hat{k}}{\sqrt{(-4)^2+4^2}} = \frac{-4\hat{j}+4\hat{k}}{4\sqrt{2}} = \frac{1}{\sqrt{2}}(-\hat{j}+\hat{k}).$$

13. Let the equation of the plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{...(i)}$$

where, a, b, c are variables.

This meets X, Y and Z axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$.

Let (α, β, γ) be the coordinates of the centroid of triangle ABC. Then,

$$\alpha = \frac{a+0+0}{3} = \frac{a}{3}, \beta = \frac{0+b+0}{3} = \frac{b}{3},$$

$$\gamma = \frac{0+0+c}{3} = \frac{c}{3} \quad \text{...(ii)}$$

The plane (i) is at a distance $3p$ from the origin.

$\therefore$ $3p$ = Length of perpendicular from (0, 0, 0) to the plane (i)

$$\Rightarrow 3p = \frac{\left|\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1\right|}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow 3p = \frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \frac{1}{9p^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad \text{...(iii)}$$

From (ii), we have

$a = 3\alpha$, $b = 3\beta$ and $c = 3\gamma$

Substituting the values of a, b, c in (iii), we get

$$\frac{1}{9p^2} = \frac{1}{9\alpha^2}+\frac{1}{9\beta^2}+\frac{1}{9\gamma^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$$

So, the locus of (α, β, γ) is $\frac{1}{p^2} = \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}$

14. (i) Clearly, according to given information, $\frac{dy}{dt} = ky(5000-y)$, where k is the constant of proportionality.

(ii) The given equation is $y = \frac{5000}{49e^{-5000kt}+1}$

Substitute $t = 0$ in the given equation, we get

$$y = \frac{5000}{49+1} \Rightarrow y = 100.$$

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