SOLUTIONS

1. We have, $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{x^2+4x+4+4}$

$= \int \frac{dx}{(x+2)^2+(2)^2} = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C$

2. The given differential equation is

$$\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 + x^4 = 0$$

Order = 2, Degree = 2

$\therefore$ Required Sum = 2 + 2 = 4

OR

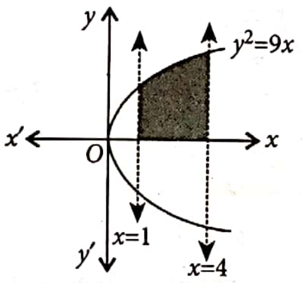
Given differential equation is $e^{\frac{dy}{dx}} = x \Rightarrow \frac{dy}{dx} = \log x$

$\Rightarrow \int dy = \int \log x \, dx + c_1$

$\Rightarrow y = x\log x - x + c_1 \Rightarrow x(\log x - 1) - y = c$

[Integrating by parts]

3. We have, $y^2 = 9x$ and lines $x = 1, x = 4$



$\therefore$ Required area

$= \int_1^4 3\sqrt{x}\,dx = 3\left[\frac{x^{3/2}}{3/2}\right]_1^4$

$= 2(4^{3/2} - 1) = 2(8 - 1)$

= 14 sq. units

4. Let $I = \int \frac{dx}{9+4x^2} = \frac{1}{4}\int \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{4}\int \frac{dx}{x^2+\left(\frac{3}{2}\right)^2}$

$= \frac{1}{4}\cdot\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)+C = \frac{1}{6}\tan^{-1}\left(\frac{2x}{3}\right)+C$

5. Here, $y = mx$ ...(i)

Differentiating (i) w.r.t. x, we get $\frac{dy}{dx} = m$ ...(ii)

Eliminating m from (i) and (ii), we get

$y = x\cdot\frac{dy}{dx} \Rightarrow x\frac{dy}{dx} - y = 0$, which is the required differential equation.

6. Given, $P(A) = 0.4$, $P(B) = 0.8$ and $P(B|A) = 0.6$.

Clearly, $P(A \cap B) = P(B|A)P(A) = 0.6 \times 0.4 = 0.24$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$= 0.4 + 0.8 - 0.24 = 0.96$

7. Let E be the event that A is coming in time.

$\therefore$ $P(E) = \frac{3}{7}$

And F be the event that B is coming in time.

$\therefore$ $P(F) = \frac{5}{7}$

Also, E and F are given to be independent events.

$\therefore$ Probability of only one of them coming to the school in time $= P(E)\cdot P(\bar{F}) + P(\bar{E})\cdot P(F)$

$= \frac{3}{7}\cdot\left(1-\frac{5}{7}\right)+\left(1-\frac{3}{7}\right)\cdot\frac{5}{7}$

$= \frac{3}{7}\cdot\frac{2}{7}+\frac{4}{7}\cdot\frac{5}{7} = \frac{26}{49}$

8. The equation of the the line AB is given by

$\frac{x-0}{4-0} = \frac{y+1}{5+1} = \frac{z+1}{1+1} = \lambda$ (say)

$\Rightarrow x = 4\lambda, y = 6\lambda - 1, z = 2\lambda - 1$

The coordinates of a general point on AB are $(4\lambda, 6\lambda - 1, 2\lambda - 1)$.

The equation of line CD is given by

$\frac{x-3}{3+4} = \frac{y-9}{9-4} = \frac{z-4}{4-4} = \mu$ (say)

$\Rightarrow x = 7\mu + 3, y = 5\mu + 9, z = 4$

The coordinates of a general point on CD are $(7\mu + 3, 5\mu + 9, 4)$

If the line AB and CD intersect, then they have a common point. So, for some values of λ and μ, we must have

$4\lambda = 7\mu + 3, 6\lambda - 1 = 5\mu + 9, 2\lambda - 1 = 4$

$\Rightarrow 4\lambda - 7\mu = 3$ ...(i), $6\lambda - 5\mu = 10$ ...(ii)

and $\lambda = \frac{5}{2}$ ...(iii)

Substituting $\lambda = \frac{5}{2}$ in (ii), we get $\mu = 1$

Since $\lambda = \frac{5}{2}$ and $\mu = 1$ satisfy (i), so the given lines AB and CD intersect.

OR

The given lines are

$\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} + 2\hat{j} + 6\hat{k})$ and

$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

S.D. between the lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is given by

$$d = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

On comparing, we get

$\vec{a}_1 = 2\hat{i} - 5\hat{j} + \hat{k}$, $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$

$\vec{a}_2 = 7\hat{i} - 6\hat{k}$, $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore\ \vec{a}_2 - \vec{a}_1 = 5\hat{i} + 5\hat{j} - 7\hat{k}$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 6 \\ 1 & 2 & 2 \end{vmatrix} = -8\hat{i} + 4\hat{k}$$

$$\therefore\ |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-8)^2 + 4^2} = 4\sqrt{5}$$

Hence, $d = \left|\dfrac{(5\hat{i} + 5\hat{j} - 7\hat{k})\cdot(-8\hat{i} + 4\hat{k})}{4\sqrt{5}}\right|$

$$= \frac{|5(-8) - 7(4)|}{4\sqrt{5}} = \frac{68}{4\sqrt{5}} = \frac{17\sqrt{5}}{5} \text{ units}$$

9. Let A be the event of drawing a red ball in first draw and B be the event of drawing a red ball in second draw.

$$\therefore\ P(A) = \frac{{}^3C_1}{{}^{10}C_1} = \frac{3}{10} \Rightarrow P(\bar{A}) = \frac{7}{10}$$

Now, $P(B/A)$ = Probability of drawing a red ball in the second draw, when a red ball already has been drawn in the first draw $= \dfrac{{}^2C_1}{{}^9C_1} = \dfrac{2}{9}$, $P\left(\dfrac{B}{\bar{A}}\right) = \dfrac{3}{9}$

Required probability = $P(A/B)$

$$= \frac{P(B/A)\cdot P(A)}{P(B/A)\cdot P(A) + P(B/\bar{A})\cdot P(\bar{A})}$$

$$= \frac{\frac{2}{9} \times \frac{3}{10}}{\frac{2}{9} \times \frac{3}{10} + \frac{3}{9} \times \frac{7}{10}} = \frac{6}{27} = \frac{2}{9}$$

10. Given, ΔABC with vertices $A(1, 1, 2)$, $B(2, 3, 5)$ and $C(1, 5, 5)$

Now, $\overrightarrow{AB} = (2-1)\hat{i} + (3-1)\hat{j} + (5-2)\hat{k}$

$= \hat{i} + 2\hat{j} + 3\hat{k}$

and $\overrightarrow{AC} = (1-1)\hat{i} + (5-1)\hat{j} + (5-2)\hat{k}$

$= 4\hat{j} + 3\hat{k}.$

$$\therefore\ (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix} = -6\hat{i} - 3\hat{j} + 4\hat{k}$$

Hence, area of $\Delta ABC = \dfrac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

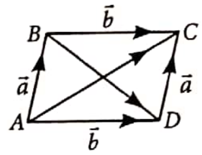
$= \dfrac{1}{2}\sqrt{(-6)^2 + (-3)^2 + 4^2}$

$= \dfrac{1}{2}\sqrt{61}$ sq. units

OR

Let $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

Then diagonal $\overrightarrow{AC}$ of the parallelogram is



$\vec{p} = \vec{a} + \vec{b}$

$= \hat{i} + 2\hat{j} + 3\hat{k} + 2\hat{i} + 4\hat{j} - 5\hat{k}$

$= 3\hat{i} + 6\hat{j} - 2\hat{k}$

Therefore unit vector parallel to it is

$$\frac{\vec{p}}{|\vec{p}|} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9 + 36 + 4}} = \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$$

Now, diagonal $\overrightarrow{BD}$ of the parallelogram is

$\vec{p}\,' = \vec{b} - \vec{a} = 2\hat{i} + 4\hat{j} - 5\hat{k} - \hat{i} - 2\hat{j} - 3\hat{k} = \hat{i} + 2\hat{j} - 8\hat{k}$

Therefore unit vector parallel to it is

$$\frac{\vec{p}\,'}{|\vec{p}\,'|} = \frac{\hat{i} + 2\hat{j} - 8\hat{k}}{\sqrt{1 + 4 + 64}} = \frac{1}{\sqrt{69}}(\hat{i} + 2\hat{j} - 8\hat{k})$$

11. Given, $\vec{a} + \vec{b} = \vec{c}$

$\Rightarrow (\vec{a} + \vec{b})\cdot(\vec{a} + \vec{b}) = \vec{c}\cdot\vec{c}$

$\Rightarrow \vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{a} = \vec{c}\cdot\vec{c}$

$\Rightarrow 1 + \vec{a}\cdot\vec{b} + 1 + \vec{a}\cdot\vec{b} = 1$

$\Rightarrow 2\vec{a}\cdot\vec{b} = -1$ ...(i)

Now, $(\vec{a} - \vec{b})^2 = (\vec{a} - \vec{b})\cdot(\vec{a} - \vec{b})$

$= \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = 1 - \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} + 1$

$= 2 - 2\vec{a}\cdot\vec{b} = 2 - (-1)$ [Using(i)]

$= 3$

$\therefore\ |\vec{a} - \vec{b}| = \sqrt{3}$

OR

Given, $|\vec{a}| = 3, |\vec{b}| = 4, |\vec{c}| = 5$ ...(i)

and $\vec{a}\cdot(\vec{b} + \vec{c}) = 0,\ \vec{b}\cdot(\vec{c} + \vec{a}) = 0,\ \vec{c}\cdot(\vec{a} + \vec{b}) = 0$

$\therefore\ \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c} + \vec{b}\cdot\vec{a} + \vec{c}\cdot\vec{a} + \vec{c}\cdot\vec{b} = 0 + 0 + 0 = 0$

$\Rightarrow 2(\vec{a}\cdot\vec{b}) + 2(\vec{b}\cdot\vec{c}) + 2(\vec{c}\cdot\vec{a}) = 0$ ...(ii)

Now, $|\vec{a}+\vec{b}+\vec{c}|^2 = (\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})$

$= (\vec{a}+\vec{b}+\vec{c})\cdot\vec{a}+(\vec{a}+\vec{b}+\vec{c})\cdot\vec{b}+(\vec{a}+\vec{b}+\vec{c})\cdot\vec{c}$

$= \vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{c}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}+\vec{c}\cdot\vec{b}+\vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{c}$

$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}\cdot\vec{b})+2(\vec{b}\cdot\vec{c})+2(\vec{c}\cdot\vec{a})$

$= 3^2+4^2+5^2+0$ [Using (i) and (ii)]

$= 50$

$\therefore |\vec{a}+\vec{b}+\vec{c}| = 5\sqrt{2}$.

12. We have, $y = 2\sin x + \sin 2x,\ 0 \le x \le 2\pi$

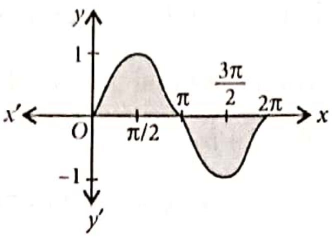
Putting $y = 0$, we get $2\sin x + \sin 2x = 0$

$\Rightarrow$ $2\sin x + 2\sin x \cos x = 0$

$\Rightarrow$ $2\sin x(1 + \cos x) = 0$

$\Rightarrow$ $\sin x = 0$ or $\cos x = -1$

$\Rightarrow$ $x = 0, \pi, 2\pi$



$\therefore$ Required area

$$= 2\int_0^{\pi}(2\sin x+\sin 2x)dx$$

$$=2\left[-2\cos x-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$=-2\left[\left(2\cos\pi+\frac{\cos 2\pi}{2}\right)-\left(2\cos 0+\frac{\cos 0}{2}\right)\right]$$

$$=-2\left[\left(-2+\frac{1}{2}\right)-\left(2+\frac{1}{2}\right)\right]=-2\left[-2+\frac{1}{2}-2-\frac{1}{2}\right]$$

$= -2[-4] = 8$ sq. units

13. Any point on the line

$$\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}=r \text{ (say)} \quad \text{...(i)}$$

is $(3r - 1, 5r - 3, 7r - 5)$.

Any point on the line

$$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}=k \text{ (say)} \quad \text{...(ii)}$$

is $(k + 2, 3k + 4, 5k + 6)$.

For lines (i) and (ii) to intersect, we must have

$3r - 1 = k + 2,\ 5r - 3 = 3k + 4,\ 7r - 5 = 5k + 6$

On solving these, we get $r=\frac{1}{2}, k=-\frac{3}{2}$

$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2},-\frac{1}{2},-\frac{3}{2}\right)$.

14. Clearly, according to given information,

$\frac{dn}{dt} = \lambda n(1000 - n)$, where λ is constant of proportionality.

(i) We have, $\frac{dn}{dt}=\lambda n(100-n)$

$$\Rightarrow \int\frac{dn}{n(1000-n)}=\lambda\int dt$$

$$\Rightarrow \frac{1}{1000}\int\left(\frac{1}{1000-n}+\frac{1}{n}\right)dn=\lambda\int dt$$

$$\Rightarrow \frac{1}{1000}\left[\frac{\log(1000-n)}{-1}+\log n\right]=\lambda t+C$$

$$\Rightarrow \frac{1}{1000}\log\left(\frac{n}{1000-n}\right)=\lambda t+C$$

(ii) Since, 50 students are infected after 4 days.

$\therefore$ $n(4) = 50$.

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