SOLUTIONS

1. We have, $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1$

or $\frac{dy}{dx}+\frac{1}{\sqrt{x}}y=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

This is a linear differential equation of the form $\frac{dy}{dx}+Py=Q$, where $P=\frac{1}{\sqrt{x}}, Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\therefore$ I.F. $= e^{\int P dx} \Rightarrow$ I.F. $= e^{\int \frac{1}{\sqrt{x}}dx} = e^{2\sqrt{x}}$

2. Given, $P(A\cup B)=\frac{3}{4}, P(A\cap B)=\frac{1}{4}$ and $P(\bar{A})=\frac{2}{3}$

Now, $P(\bar{A})=\frac{2}{3} \Rightarrow 1-P(A)=\frac{2}{3} \Rightarrow P(A)=\frac{1}{3}$.

Now, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$\Rightarrow \frac{3}{4}=\frac{1}{3}+P(B)-\frac{1}{4} \Rightarrow P(B)=\frac{3}{4}+\frac{1}{4}-\frac{1}{3}=\frac{2}{3}$.

3. Let $I=\int_{-\pi/4}^{0}\frac{(1+\tan x)}{(1-\tan x)}dx=\int_{-\pi/4}^{0}\frac{\left(1+\frac{\sin x}{\cos x}\right)}{\left(1-\frac{\sin x}{\cos x}\right)}dx$

$=\int_{-\pi/4}^{0}\frac{\cos x+\sin x}{\cos x-\sin x}dx$

Put $\cos x - \sin x = t \Rightarrow -(\sin x + \cos x)\,dx = dt$

When $x=0, t=1$ and when $x=\frac{-\pi}{4}, t=\sqrt{2}$

$\therefore\ I=\int_{\sqrt{2}}^{1}-\frac{dt}{t}=\int_{1}^{\sqrt{2}}\frac{dt}{t}=[\log t]_1^{\sqrt{2}}$

$=\log\sqrt{2}-\log 1=\frac{1}{2}\log 2$

4. Given, $P(A)=\frac{2}{5}, P(B)=\frac{3}{10}, P(A\cap B)=\frac{1}{5}$

Now, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$=\frac{2}{5}+\frac{3}{10}-\frac{1}{5}=\frac{1}{2}$

$\therefore\ P(A'\cap B')=P((A\cup B)')=1-P(A\cup B)=1-\frac{1}{2}=\frac{1}{2}$

Also, $P(B')=1-P(B)=1-\frac{3}{10}=\frac{7}{10}$

$\therefore\ P(A'|B')=\frac{P(A'\cap B')}{P(B')}=\frac{1/2}{7/10}=\frac{5}{7}$

OR

We have, $P(A) = 1/4$

Now, $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$= P(A) + P(B) - P(A)\,P(B)$ ($\because$ A, B are independent)

$= 1/4 + P(B) - (1/4)\,P(B) = 2P(B) - 1/4$ (Given)

$\Rightarrow P(B) = 2/5$

5. The given differential equation is

$x^2\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^4$

$\therefore$ Its order is 2 and degree is 1.

6. Since, area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab.

$\therefore$ Required area $= \pi\times 4\times 9 = 36\pi$ sq. units.

7. Let $I=\int\frac{5x+3}{\sqrt{x^2+4x+10}}dx=\int\frac{\frac{5}{2}(2x+4)-7}{\sqrt{x^2+4x+10}}dx$

$=\frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx-7\int\frac{dx}{\sqrt{x^2+4x+10}}$

$= I_1 + I_2$ (say) ...(1)

where $I_1=\frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$

Put $x^2 + 4x + 10 = t \Rightarrow (2x + 4)dx = dt$

$\therefore\ I_1=\frac{5}{2}\int t^{-1/2}dt=\frac{5}{2}\cdot\frac{t^{1/2}}{(1/2)}=5\sqrt{t}$

$=5\sqrt{x^2+4x+10}+C_1$ ...(2)

and $I_2=-7\int\frac{dx}{\sqrt{x^2+4x+10}}$

$=-7\int\frac{dx}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$

$=-7\log|x+2+\sqrt{x^2+4x+10}|+C_2$ ...(3)

From (1), (2) and (3), we get

$I=5\sqrt{x^2+4x+10}-7\log|x+2+\sqrt{x^2+4x+10}|+C$,

where $C = C_1 + C_2$

8. We have, $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$

Let $\vec{r}=\vec{a}+\vec{b}=2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{p}=\vec{a}-\vec{b}=-\hat{j}-2\hat{k}$

A unit vector perpendicular to both $\vec{r}$ and $\vec{p}$ is given as $\pm\frac{\vec{r}\times\vec{p}}{|\vec{r}\times\vec{p}|}$.

Now, $\vec{r}\times\vec{p}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 0 & -1 & -2\end{vmatrix}=-2\hat{i}+4\hat{j}-2\hat{k}$

So, the required unit vector is

$$=\pm\frac{(-2\hat{i}+4\hat{j}-2\hat{k})}{\sqrt{(-2)^2+4^2+(-2)^2}}=\mp\frac{(\hat{i}-2\hat{j}+\hat{k})}{\sqrt{6}}.$$

9. The given line is $\frac{x+2}{2}=\frac{2y-7}{6}=\frac{5-z}{6}$

$$\Rightarrow \frac{x+2}{2}=\frac{y-\frac{7}{2}}{3}=\frac{z-5}{-6} \quad \text{...(i)}$$

Its d.r's are 2, 3, –6

$\because \sqrt{2^2+3^2+(-6)^2}=7$

$\therefore$ Its d.c's are $\frac{2}{7},\frac{3}{7},-\frac{6}{7}$

Eq. of a line through (–1, 2, 3) and parallel to (i) is

$$\frac{x+1}{2}=\frac{y-2}{3}=\frac{z-3}{-6}=\lambda \text{ (say)}$$

$\therefore$ Vector equation of a line passing through (–1, 2, 3) and parallel to (i) is given by

$$\vec{r}=(-\hat{i}+2\hat{j}+3\hat{k})+\lambda(2\hat{i}+3\hat{j}-6\hat{k})$$

OR

Eq. of any plane through (–1, 2, 0) is

$a(x+1)+b(y-2)+cz=0$ ...(i)

Also it passes through (2, 2, –1)

$\therefore\ 3a+0\cdot b-c=0$ ...(ii)

Further plane (i) is parallel to the line

$$\frac{x-1}{1}=\frac{y+\frac{1}{2}}{1}=\frac{z+1}{-1}$$

$\therefore\ a\cdot 1+b\cdot 1+c\cdot(-1)=0$

$\Rightarrow a+b-c=0$ ...(iii)

From (ii) and (iii), we get

$\frac{a}{1}=\frac{b}{2}=\frac{c}{3}=\lambda(\text{say})$

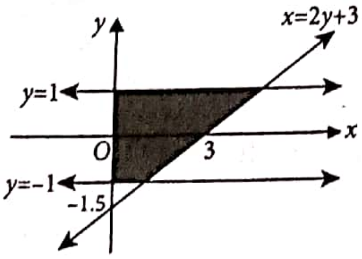
$\Rightarrow a=\lambda,\ b=2\lambda$ and $c=3\lambda$

Put these values in (i), we get

$x+2y+3z=3$

This is the eq. of the required plane.

10. We have, $x=2y+3$, a straight line



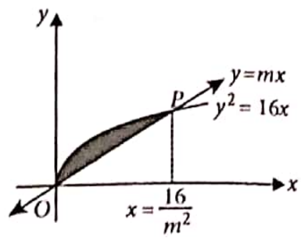
$\therefore$ Required area

$$=\int_{-1}^{1}(2y+3)\,dy=\left[y^2+3y\right]_{-1}^{1}$$

$=(1+3)-(1-3)=4+2=6$ sq. units

OR

We have, $y^2=16x$, a parabola with vertex (0, 0) and line $y=mx$.



$\therefore$ Required area

$$=\int_0^{16/m^2}(\sqrt{16x}-mx)\,dx=\frac{2}{3}$$

$$\Rightarrow \left[4\times\frac{2}{3}x^{3/2}-m\left(\frac{x^2}{2}\right)\right]_0^{16/m^2}=\frac{2}{3}$$

$$\Rightarrow \frac{8}{3}\times\frac{64}{m^3}-\frac{m}{2}\frac{256}{m^4}=\frac{2}{3}\Rightarrow\frac{1}{m^3}\left[\frac{512}{3}-128\right]=\frac{2}{3}$$

$\Rightarrow m^3=64 \Rightarrow m=4$

11. Let A, E_1 and E_2 respectively be the events that a person has a heart attack, the selected person followed the course of yoga and meditation and the person adopted the drug prescription.

$$P(A)=\frac{40}{100}=0.40,\ P(E_1)=P(E_2)=\frac{1}{2}$$

$P(A/E_1)=0.40\times0.70=0.28,$

$P(A/E_2)=0.40\times0.75=0.30$

Probability that the patient suffering from heart attack followed the course of meditation and yoga is

$$P(E_1/A)=\frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\frac{\frac{1}{2}\times0.28}{\frac{1}{2}\times0.28+\frac{1}{2}\times0.30}=\frac{0.14}{0.14+0.15}=\frac{14}{29}$$

Now, $$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\frac{\frac{1}{2}\times0.30}{\frac{1}{2}\times0.28+\frac{1}{2}\times0.30}=\frac{0.15}{0.14+0.15}=\frac{15}{29}$$

12. Here $\vec{a}=3\hat{i}-\hat{j},\ \vec{b}=2\hat{i}+\hat{j}-3\hat{k}$

We have to express : $\vec{b}=\vec{b}_1+\vec{b}_2$, where

$\vec{b}_1\,||\,\vec{a}$ and $\vec{b}_2\perp\vec{a}$

Let $\vec{b}_1=\lambda\vec{a}=\lambda(3\hat{i}-\hat{j})$ and $\vec{b}_2=x\hat{i}+y\hat{j}+z\hat{k}$

Now $\vec{b}_2 \perp \vec{a} \Rightarrow \vec{b}_2 \cdot \vec{a} = 0$

$\Rightarrow (x\hat{i}+y\hat{j}+z\hat{k})\cdot(3\hat{i}-\hat{j})=0$

$\Rightarrow 3x - y = 0$ ...(i)

Now, $\vec{b} = \vec{b}_1 + \vec{b}_2$

$\Rightarrow 2\hat{i}+\hat{j}-3\hat{k} = \lambda(3\hat{i}-\hat{j})+(x\hat{i}+y\hat{j}+z\hat{k})$

On comparing, we get

$\left.\begin{array}{l}2=3\lambda+x\\1=-\lambda+y\end{array}\right]\Rightarrow x+3y=5$ ...(ii)

and $-3 = z \Rightarrow z = -3$

Solving (i) and (ii), we get $x=\frac{1}{2}, y=\frac{3}{2}$

$\therefore 1=-\lambda+y \Rightarrow 1=-\lambda+\frac{3}{2} \Rightarrow \lambda=\frac{1}{2}$

Hence, $\vec{b}_1 = \lambda(3\hat{i}-\hat{j}) = \frac{3}{2}\hat{i}-\frac{1}{2}\hat{j}$

and $\vec{b}_2 = \frac{1}{2}\hat{i}+\frac{3}{2}\hat{j}-3\hat{k}$

OR

Given, position vector of $A = \hat{i}+\hat{j}+\hat{k}$

Position vector of $B = 2\hat{i}+5\hat{j}$

Position vector of $C = 3\hat{i}+2\hat{j}-3\hat{k}$

Position vector of $D = \hat{i}-6\hat{j}-\hat{k}$

$\therefore \overline{AB} = (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$ and

$\overline{CD} = (\hat{i}-6\hat{j}-\hat{k})-(3\hat{i}+2\hat{j}-3\hat{k}) = -2\hat{i}-8\hat{j}+2\hat{k}$

Now $|\overline{AB}| = \sqrt{(1)^2+(4)^2+(1)^2} = \sqrt{18}$

$|\overline{CD}| = \sqrt{(-2)^2+(-8)^2+(2)^2} = \sqrt{4+64+4}$

$= \sqrt{72} = 2\sqrt{18}$

Let θ be the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

$\therefore \cos\theta = \frac{\overline{AB}\cdot\overline{CD}}{|\overline{AB}||\overline{CD}|} = \frac{(\hat{i}+4\hat{j}-\hat{k})\cdot(-2\hat{i}-8\hat{j}+2\hat{k})}{(\sqrt{18})(2\sqrt{18})}$

$= \frac{-2-32-2}{36} = \frac{-36}{36} = -1$

$\Rightarrow \cos\theta = -1 \Rightarrow \theta = \pi$

Since, angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ is 180°.

$\therefore$ $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

13. The given line is

$\vec{r} = -\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$

Its cartesian eq. is

$\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1} = \lambda(\text{say})$ ...(i)

Any point Q on (i) is $(2\lambda - 1, 3\lambda + 3, -\lambda + 1)$

Also, the given point is $P(5, 4, 2)$.

Now d.r's of the line PQ are

$(2\lambda-1-5, 3\lambda+3-4, -\lambda+1-2) = (2\lambda-6, 3\lambda-1, -\lambda-1)$.

For PQ to be $\perp$ to (i), we must have

$(2\lambda-6)\cdot 2 + (3\lambda-1)\cdot 3 + (-\lambda-1)\cdot(-1) = 0$

$\Rightarrow 14\lambda - 14 = 0 \Rightarrow \lambda = 1$

$\therefore$ Q is $(1, 6, 0)$, which is the foot of $\perp$ from P on line (i).

Now, $PQ = \sqrt{(5-1)^2+(4-6)^2+(2-0)^2}$

$= \sqrt{24} = 2\sqrt{6}$ units.

Further if $R(\alpha, \beta, \gamma)$ is the image of P in line (i), then

$\frac{\alpha+5}{2} = 1, \frac{\beta+4}{2} = 6, \frac{\gamma+2}{2} = 0$

$\Rightarrow \alpha = -3, \beta = 8, \gamma = -2$

$\therefore$ Image of P in line (i) is $R(-3, 8, -2)$.

14. (i) We have, $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{x^2+x\cdot vx+v^2x^2}{x^2} = 1+v+v^2$

$\Rightarrow x\frac{dv}{dx} = 1+v^2 \Rightarrow \int\frac{dv}{1+v^2} = \int\frac{dx}{x}$

$\Rightarrow \tan^{-1}v = \log|x|+c \Rightarrow \tan^{-1}\frac{y}{x} = \log|x|+c$

(ii) We have, $2xy\frac{dy}{dx} = x^2+3y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$

Put $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore \quad v + x\frac{dv}{dx} = \frac{x^2+3v^2x^2}{2vx^2} \Rightarrow x\frac{dv}{dx} = \frac{1+3v^2}{2v} - v$

$\Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{2v} \Rightarrow \int\frac{2v}{1+v^2}dv = \int\frac{dx}{x}$

$\Rightarrow \log|1+v^2| = \log|x| + \log|c| \Rightarrow \log|v^2+1| = \log|xc|$

$\Rightarrow v^2+1 = xc \Rightarrow \frac{y^2}{x^2}+1 = xc \Rightarrow x^2+y^2 = x^3c$

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