SOLUTIONS

1. The given differential equation has the order of the highest order derivative as 3 and its power is 2. Hence, its order is 3 and degree is 2.

OR

We have, $5\frac{dy}{dx} = e^x y^4 \Rightarrow 5\frac{dy}{y^4} = e^x dx$

On Integrating both sides, we get

$\Rightarrow 5.\int y^{-4}dy = \int e^x dx$

$\Rightarrow 5.\frac{y^{-3}}{(-3)} = e^x + c \Rightarrow \frac{-5}{3y^3} = e^x + c$

2. If A and B are two independent events, then $P(A \cap B) = P(A) \times P(B)$

It is given that $P(A \cup B) = 0.6, P(A) = 0.2$

$\therefore \; P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \; P(A \cup B) = P(A) + P(B) - P(A) \times P(B)$

$\Rightarrow \; 0.6 = 0.2 + P(B)(1 - 0.2) \;\; \Rightarrow \;\; 0.4 = P(B)0.8$

$\Rightarrow \; P(B) = \frac{0.4}{0.8} = \frac{1}{2}.$

3. Consider the following events :

A = Selecting a fair complexioned student;

B = Selecting a rich student;

C = Selecting a girl.

We have, $P(A) = \frac{20}{80} = \frac{1}{4}$, $P(B) = \frac{10}{80} = \frac{1}{8}$ and

$P(C) = \frac{25}{80} = \frac{5}{16}$

Since A, B, C are independent events, therefore,

Required probability $= P(A \cap B \cap C)$

$= P(A)\, P(B)\, P(C)$

$= \frac{1}{4} \times \frac{1}{8} \times \frac{5}{16} = \frac{5}{512}$

4. Let $I = \int \frac{dx}{\sqrt{2+4x-x^2}} = \int \frac{dx}{\sqrt{2+4-4+4x-x^2}}$

$\Rightarrow \; I = \int \frac{dx}{\sqrt{6-(x^2-4x+4)}} = \int \frac{dx}{\sqrt{(\sqrt{6})^2-(x-2)^2}}$

$= \sin^{-1}\frac{(x-2)}{\sqrt{6}} + C$

5. Let $I = \int_1^2 \left[\frac{1}{x} - \frac{1}{2x^2}\right] e^{2x} dx$

Putting $2x = y \Rightarrow 2dx = dy$

As $x \to 1 \Rightarrow y \to 2$ and $x \to 2 \Rightarrow y \to 4$

$\therefore \; I = \frac{1}{2}\int_2^4 \left[\frac{2}{y} - \frac{2}{y^2}\right] e^y dy = \int_2^4 \left[\frac{1}{y} - \frac{1}{y^2}\right] e^y dy$

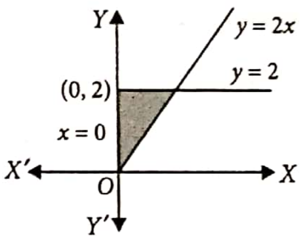
$= \left[e^y \cdot \frac{1}{y}\right]_2^4 = \frac{1}{4}e^4 - \frac{1}{2}e^2 = \frac{e^2}{2}\left(\frac{e^2}{2} - 1\right)$

6. Given curve is $y = 2x$

$\therefore$ Required area $= \int_0^2 \frac{y}{2} dy$

$= \left[\frac{y^2}{4}\right]_0^2$

$= \frac{4}{4} = 1$ sq. unit



7. Let $I = \int \frac{\cos x}{\sin^2 x + 4\sin x + 5} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$\therefore \; I = \int \frac{dt}{t^2 + 4t + 5} = \int \frac{dt}{t^2 + 4t + 4 + 1}$

$\Rightarrow \; I = \int \frac{dt}{(t+2)^2 + (1)^2} = \tan^{-1}(t+2) + C$

$\Rightarrow \; I = \tan^{-1}(\sin x + 2) + C$

OR

Let $I = \int \sqrt{\frac{a-x}{a+x}} dx$

$\Rightarrow \; I = \int \sqrt{\frac{(a-x)\times(a-x)}{(a+x)\times(a-x)}} dx = \int \frac{a-x}{\sqrt{a^2-x^2}} dx$

$\Rightarrow \; I = \int \frac{a}{\sqrt{a^2-x^2}} dx - \int \frac{x}{\sqrt{a^2-x^2}} dx$

$\left[\text{Put } a^2 - x^2 = t \;\; \Rightarrow \;\; -2x dx = dt, \Rightarrow -x\,dx = \frac{1}{2}dt\right]$

$I = a\sin^{-1}\frac{x}{a} + \frac{1}{2}\int \frac{dt}{\sqrt{t}} = a\sin^{-1}\frac{x}{a} + \sqrt{t} + C$

$\Rightarrow \; I = a\sin^{-1}\frac{x}{a} + \sqrt{a^2 - x^2} + C$

8. We have, $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{c} = 0$

$\Rightarrow \vec{a} \perp \vec{b}$ and $\vec{a} \perp \vec{c}$

$\Rightarrow$ $\vec{a}$ is perpendicular to the plane of $\vec{b}$ and $\vec{c}$.

$\Rightarrow$ $\vec{a}$ is parallel to $\vec{b} \times \vec{c}$

$\Rightarrow$ $\vec{a} = \lambda(\vec{b} \times \vec{c})$ for some scalar λ.

$\Rightarrow\ |\vec{a}|=|\lambda||\vec{b}\times\vec{c}| \Rightarrow |\vec{a}|=|\lambda||\vec{b}||\vec{c}|\sin\frac{\pi}{6}$

$\Rightarrow\ 1=\frac{|\lambda|}{2}$ $\quad [\because |\vec{a}|=|\vec{b}|=|\vec{c}|=1]$

$\Rightarrow\ |\lambda|=2 \Rightarrow \lambda=\pm 2$

$\therefore\ \vec{a}=\lambda(\vec{b}\times\vec{c}) \Rightarrow \vec{a}=\pm 2(\vec{b}\times\vec{c})$

OR

We have, $\vec{a}=5\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\lambda\hat{k}$

$\Rightarrow \vec{a}+\vec{b}=(5\hat{i}-\hat{j}+7\hat{k})+(\hat{i}-\hat{j}+\lambda\hat{k})=6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$

and $\vec{a}-\vec{b}=(5\hat{i}-\hat{j}+7\hat{k})-(\hat{i}-\hat{j}+\lambda\hat{k})=4\hat{i}+(7-\lambda)\hat{k}$

Since, $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are orthogonal.

$\therefore\ (\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$

$\Rightarrow\ (6\hat{i}-2\hat{j}+(7+\lambda)\hat{k})\cdot(4\hat{i}+(7-\lambda)\hat{k})=0$

$\Rightarrow\ 24+49-\lambda^2=0 \Rightarrow \lambda^2=73 \Rightarrow \lambda=\pm\sqrt{73}$

9. The equation of a plane through the intersection of $\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})=5$ and $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})=3$ is

$[\vec{r}\cdot(\hat{i}+3\hat{j}-\hat{k})-5]+\lambda[\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})-3]=0$

$\Rightarrow\ \vec{r}\cdot[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}]-(5+3\lambda)=0$ ...(i)

If plane in (i) passes through (2, 1, – 2), then the vector $2\hat{i}+\hat{j}-2\hat{k}$ should satisfy it.

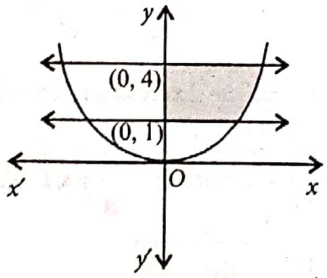
$\therefore\ (2\hat{i}+\hat{j}-2\hat{k})\cdot[(1+2\lambda)\hat{i}+(3-\lambda)\hat{j}+(-1+\lambda)\hat{k}]-(5+3\lambda)=0$

$\Rightarrow\ 2(1+2\lambda)+1(3-\lambda)-2(-1+\lambda)-(5+3\lambda)=0$

$\Rightarrow\ -2\lambda+2=0 \Rightarrow \lambda=1$

Putting $\lambda=1$ in (i), we get the required equation of the plane as $\vec{r}\cdot(3\hat{i}+2\hat{j}+0\hat{k})=8$.

10. The rough sketch of the curve $y=9x^2, x=0, y=1$ and $y=4$ is as shown in the figure.



The required area of shaded region $=\int_1^4 x\,dy$

$=\int_1^4 \frac{\sqrt{y}}{3}dy=\frac{1}{3}\left[\frac{2}{3}(y)^{3/2}\right]_1^4$

$=\frac{2}{9}\left[4^{3/2}-1\right]=\frac{2}{9}(8-1)=\frac{14}{9}$ sq.units

11. Let E_1 and E_2 be the events of choosing a bicycle from the first plant and the second plant respectively. Let E be the event of choosing a bicycle of standard quality. Then,

$P(E_1)=\frac{60}{100}=\frac{3}{5}$, and $P(E_2)=\frac{40}{100}=\frac{2}{5}$.

$P(E|E_1)$ = probability of choosing a bicycle of standard quality, given that it is produced by the first plant

$=\frac{80}{100}=\frac{4}{5}$.

$P(E|E_2)$ = probability of choosing a bicycle of standard quality, given that it is produced by the second plant

$=\frac{90}{100}=\frac{9}{10}$.

$\therefore$ Required probability

$= P(E_2|E)$ = probability of choosing a bicycle from the second plant, given that it is of standard quality

$=\frac{P(E_2)\cdot P(E|E_2)}{P(E_1)\cdot P(E|E_1)+P(E_2)\cdot P(E|E_2)}$ [By Bayes' theorem]

$=\frac{\left(\frac{2}{5}\times\frac{9}{10}\right)}{\left(\frac{3}{5}\times\frac{4}{5}\right)+\left(\frac{2}{5}\times\frac{9}{10}\right)}=\frac{3}{7}$.

12. We have,

$\vec{a}=4\hat{i}+5\hat{j}-\hat{k}, \vec{b}=\hat{i}-4\hat{j}+5\hat{k}, \vec{c}=3\hat{i}+\hat{j}-\hat{k}$

Since $\vec{d}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, therefore it is parallel to $\vec{a}\times\vec{b}$. So, let $\vec{d}=\lambda(\vec{a}\times\vec{b})$

$\Rightarrow\ \vec{d}=\lambda\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 5 & -1\\ 1 & -4 & 5\end{vmatrix}$

$=\lambda[(25-4)\hat{i}-(20+1)\hat{j}+(-16-5)\hat{k}]$

$\Rightarrow\ \vec{d}=\lambda(21\hat{i}-21\hat{j}-21\hat{k})$ ...(i)

Now, $\vec{c}\cdot\vec{d}=21$

$\Rightarrow\ (3\hat{i}+\hat{j}-\hat{k})\cdot\lambda(21\hat{i}-21\hat{j}-21\hat{k})=21$

$\Rightarrow\ 21\lambda(3-1+1)=21 \Rightarrow \lambda=\frac{1}{3}$

Putting $\lambda=\frac{1}{3}$ in (i), we get

$\vec{d}=\frac{1}{3}(21\hat{i}-21\hat{j}-21\hat{k})=7\hat{i}-7\hat{j}-7\hat{k}$

13. The equation of a plane passing through (1, 0, –1) is $a(x-1)+b(y-0)+c(z+1)=0$ ...(i)

This passes through (3, 2, 2).

$\therefore \quad a(3-1)+b(2-0)+c(2+1)=0$

$\Rightarrow \quad 2a+2b+3c=0$ ...(ii)

The plane in (i) is parallel to the line

$$\frac{x-1}{1}=\frac{y-1}{-2}=\frac{z-2}{3}.$$

Therefore, normal to the plane is perpendicular to the line.

$\therefore \quad a(1)+b(-2)+c(3)=0 \Rightarrow a-2b+3c=0$ ...(iii)

Solving (ii) and (iii) by cross-multiplication, we get

$$\frac{a}{(2)(3)-(3)(-2)}=\frac{b}{(1)(3)-(2)(3)}=\frac{c}{(2)(-2)-(2)(1)}$$

$$\Rightarrow \frac{a}{12}=\frac{b}{-3}=\frac{c}{-6} \Rightarrow \frac{a}{4}=\frac{b}{-1}=\frac{c}{-2}=\lambda \text{ (say)}$$

$\Rightarrow \quad a=4\lambda, b=-\lambda, c=-2\lambda$

Substituting the values of a, b, c in (i), we obtain

$4x-y-2z-6=0$

This is the equation of required plane.

OR

The equation of two given lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad \text{...(i)}$$

and $$\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5} \quad \text{...(ii)}$$

Line (i) passes through (1, 2, 3) and has direction ratios proportional to 2, 3, 4. So, its vector equation is

$\vec{r}=\vec{a}_1+\lambda\vec{b}_1$ ...(iii)

where, $\vec{a}_1=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{b}_1=2\hat{i}+3\hat{j}+4\hat{k}$

Line (ii) passes through (2, 4, 5) and has direction ratios proportional to 3, 4, 5. So, its vector equation is

$\vec{r}=\vec{a}_2+\mu\vec{b}_2$ ...(iv)

where, $\vec{a}_2=2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{b}_2=3\hat{i}+4\hat{j}+5\hat{k}$

The shortest distance between the lines (iii) and (iv) is given by

$$\text{S.D.}=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right| \quad \text{...(v)}$$

We have,

$$\vec{a}_2-\vec{a}_1=(2\hat{i}+4\hat{j}+5\hat{k})-(\hat{i}+2\hat{j}+3\hat{k})=\hat{i}+2\hat{j}+2\hat{k}$$

and $$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 3 & 4 & 5\end{vmatrix}=-\hat{i}+2\hat{j}-\hat{k}$$

$\therefore \quad |\vec{b}_1\times\vec{b}_2|=\sqrt{1+4+1}=\sqrt{6}$

and $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)=(\hat{i}+2\hat{j}+2\hat{k})\cdot(-\hat{i}+2\hat{j}-\hat{k})$

$=-1+4-2=1$

Substituting the values of $(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)$ and $|\vec{b}_1\times\vec{b}_2|$ in (v), we obtain S.D. $=1/\sqrt{6}$ unit.

14. Here, P denotes the principal at any time t and the rate of interest be $r\%$ per annum compounded continuously, then according to the law given in the problem, we get

$$\frac{dP}{dt}=\frac{Pr}{100}$$

(i) We have, $\frac{dP}{dt}=\frac{Pr}{100}$

$$\Rightarrow \frac{dP}{P}=\frac{r}{100}dt \Rightarrow \int\frac{1}{P}dP=\frac{r}{100}\int dt$$

$$\Rightarrow \log P=\frac{rt}{100}+C \quad \text{...(1)}$$

At $t=0$, $P=P_0$.

$\therefore \quad C=\log P_0$

So, $\log P=\frac{rt}{100}+\log P_0$

$$\Rightarrow \log\left(\frac{P}{P_0}\right)=\frac{rt}{100} \quad \text{...(2)}$$

(ii) We have, $r=5$, $P_0=$ ₹ 100 and $P=$ ₹ $200=2P_0$

Substituting these values in (2), we get

$$\log 2=\frac{5}{100}t$$

$\Rightarrow \quad t=20\log_e 2=20\times 0.6931$ years $=13.862$ years

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